Classification of Abelian and Non-Abelian Multilayer Fractional Quantum Hall States Through the Pattern of Zeros

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A large class of fractional quantum Hall (FQH) states can be classified according to their pattern of zeros, which describes the way ideal ground state wave functions go to zero as various clusters of electrons are brought together. In this paper we generalize this approach to classify multilayer FQH states. Such a classification leads to the construction of a class of non-Abelian multilayer FQH states that are closely related to $\hat{g}_k$ parafermion conformal field theories, where $\hat{g}_k$ is an affine simple Lie algebra. We discuss the possibility of some of the simplest of these non-Abelian states occuring in experiments on bilayer FQH systems at $\nu = 2/3$, $4/5$, $4/7$, etc.

I. INTRODUCTION

One of the most important questions in condensed matter physics relates to how we should characterize and classify the many different quantum phases of matter. A large part of the story is the theory of symmetry breaking, which tells us that we should classify various phases based on the symmetries of the ground state wave function. Yet with the discovery of the fractional quantum Hall (FQH) states[1,2] came also the understanding that there are many distinct and fascinating quantum phases of matter, called topological phases, whose characterization has nothing at all to do with symmetry. How should we characterize and systematically classify the different possible topological phases that may occur in a FQH system?

Let us run through a few obvious possibilities. We know that the FQH states contain topology-dependent and topologically stable degenerate ground states, which allow us to introduce the concept of topological order in FQH states.[3,4] Such topology-dependent degenerate ground states suggest that the low energy theories describing the FQH states are topological gauge field theories in 2+1 dimensions, called Chern-Simons theories.[5–8] So one possibility is that we may try to classify the different FQH phases by classifying all of the different possible Chern-Simons theories. But this is not a satisfactory approach for non-Abelian FQH states because we do not have a good way of knowing which Chern-Simons theories can possibly correspond to a physical system made of electrons and which cannot.

Another possibility comes through the connection between FQH wavefunctions and conformal field theory (CFT). It was discovered around 1990 that correlation functions in certain two-dimensional conformal field theories may serve as good model wavefunctions for FQH states.[9] Thus perhaps we may classify FQH states by classifying all of the different conformal field theories that may be used to construct a "valid" FQH wavefunction. While the connection between FQH wave functions and CFT correlators has been extremely fruitful in both constructing new model wavefunctions and understanding their topological properties, the CFT approach is incomplete in that there does not exist a complete classification of all possible conformal field theories that may be used to construct FQH wavefunctions. This precludes the possibility of systematically classifying FQH states. The CFT approach also obscures the essential physics and the essential properties of the conformal field theory that allow one to obtain amenable FQH wavefunctions.

In an attempt to obtain a systematic classification of FQH states, it was shown recently that a wide class of single-component FQH states and their topological excitations can be classified by their pattern of zeros, which describe the way ideal FQH wavefunctions go to zero when various clusters of particles are brought together.[10–12] This analysis led to the discovery of some new non-Abelian FQH states whose corresponding CFT has not yet been identified. It also helped elucidate the role of CFT in constructing FQH wavefunctions. The CFT encodes the way the wavefunction goes to zero as various clusters of electrons are brought together. The order of these zeros must satisfy certain conditions and the solutions to these conditions correspond to particular CFTs. Thus in classifying and characterizing FQH states, one can bypass the CFT altogether and proceed directly to classifying the different allowed pattern of zeros and subsequently obtaining the topological properties of the quasiparticles from the pattern of zeros. This construction can then even be thought of as a classification of the allowed CFTs that can be used to construct FQH states. Furthermore, these considerations give way to a natural notion of which pattern of zeros solutions are simpler than other ones. In this sense, then, one can see that the Moore-Read Pfaffian quantum Hall state[9] is the "simplest" non-Abelian generalization of the Laughlin state.

In this paper, we generalize the pattern-of-zeros classification to multilayer FQH wave functions. Such a systematic classification leads us to the construction of non-Abelian multilayer FQH states and also helps identify the simplest non-Abelian generalizations of the Halperin[13] bilayer FQH states. Therefore, in the same way that the Pfaffian FQH state is the simplest non-Abelian single-layer state and so one of the first non-Abelian states expected to be realized experimentally, our construction allows us to identify the simplest non-Abelian bilayer states and therefore some of the ones that deserve further con-

sideration in future experimental and numerical work.

We would like to point out that the "1D charge-density-wave" characterization of single-component FQH states[14,15] is closely related to the pattern-of-zeros approach. Our pattern-of-zeros approach for multilayer FQH states can also be viewed as a way to generalize the "1D charge-density-wave" characterization to multi-component cases.[16]

This paper is organized as follows. We begin by describing the ideal FQH wave functions that we can characterize by the pattern of zeros and their relation to symmetric holomorphic polynomials. In Section IV, we describe the different ways to characterize the pattern of zeros. In Section V we find the conditions that the pattern of zeros must satisfy in order to describe valid FQH wave functions. In Section VI, we sketch how one may begin to construct ideal Hamiltonians whose ground states will be FQH wave functions with a given pattern of zeros. After a brief summary of the pattern of zeros data and conditions in Section VII, we explain in Section VIII the relation between the pattern-of-zeros approach and the CFT approach to FQH wave functions. In Section IX, we describe some example solutions of this systematic classification of multilayer FQH wave functions, which yields many non-Abelian multilayer states. In Section X we discuss some of the simplest of these non-Abelian bilayer states that may be relevant for experiments on two-component quantum Hall systems and that warrant further numerical study.

II. FRACTIONAL QUANTUM HALL STATES AND SYMMETRIC POLYNOMIALS

The ground state wave function of a two-dimensional system of electrons in the lowest Landau level can be written in the form

$$\Psi = \Phi(z_1, \cdots, z_{N_e}) e^{-\frac{1}{4}\sum_{i=1}^{N_e} |z_i|^2}, \tag{1}$$

where $z_i = x_i + iy_i$, (x_i, y_i) are the coordinates of the i^{th} electron, and $\Phi(z_1, \cdots, z_N)$ is a holomorphic function of z_i. Since the electrons obey Fermi statistics, Φ is antisymmetric under interchange of any two coordinates z_i and z_j when all of the electrons are identical. In many physical situations, the electrons may be distinguished by various quantum numbers, such as a spin index (when the Zeeman energy is not too high), a layer index (in a multilayer two-dimensional electron system), a valley index (such as in graphene and SiGe heterostructures), *etc* . In such cases, the ground state wave function in the lowest Landau level may instead be written in the form

$$\Psi = \Phi(\{z_i^I\}) e^{-\frac{1}{4}\sum_{i,I} |z_i^I|^2}, \tag{2}$$

where $I = 1, \cdots, N_f$ is a flavor index and N_f is the number of different flavors. Φ is then necessarily antisymmetric only under interchange of z_i^I and z_j^I for any i and j. Given any antisymmetric polynomial $\Phi_{\text{anti-sym}}(\{z_i^I\})$, we can uniquely construct a symmetric polynomial:

$$\Phi_{\text{sym}}(\{z_i^I\}) = \frac{\Phi_{\text{anti-sym}}(\{z_i^I\})}{\prod_{I;i<j}(z_i^I - z_j^I)}. \tag{3}$$

Φ_{symm} will also be a polynomial because $\Phi_{\text{anti-sym}}$ must vanish when any two identical particles approach each other. Thus the above division by the factor $\prod_{I;i<j}(z_i^I - z_j^I)$ will never produce any poles in the resulting function.

Therefore to classify FQH phases of electrons, we can restrict our attention mainly to symmetric polynomials $\Phi(\{z_i^I\})$, where Φ is invariant under the interchange of z_i^I and z_j^I for any i and j, but not necessarily invariant under the interchange of z_i^I and z_j^J if $I \neq J$. In this paper we will often refer to I as a layer index. In the following, Φ will always refer to such a symmetric multilayer polynomial.

We will introduce data, such as n, m, and $S_{\vec{a}}$, to characterize and classify bosonic FQH states (*ie* symmetric polynomials) $\Phi(\{z_i^I\})$. From the above discussion, we see that the same set of data also characterizes fermionic FQH states whose wave functions are given by $\Phi_{\text{anti-symm}}(\{z_i^I\}) = \Phi(\{z_i^I\}) \prod_{I;i<j}(z_i^I - z_j^I)$.

III. IDEAL HAMILTONIANS AND IDEAL WAVE FUNCTIONS

Progress in understanding the various possible topological phases in the FQH effect has occurred largely because of the discovery of various kinds of model wave functions and the ideal Hamiltonians that yield these wavefunctions as their ground states. In the pattern of zeros approach, we classify all of the possible ideal wave functions for FQH phases. In this section we will explain what we mean by "ideal."

For each topological phase in the FQH system, we want to have a certain representative wave function that captures the topological properties of the phase. The prototypical example of such an ideal wavefunction is the Laughlin wave function at filling fraction $\nu = 1/m$:

$$\Phi_{1/m}(\{z_i\}) = \prod_{i<j}(z_i - z_j)^m. \tag{4}$$

At every particle in $\Phi_{1/m}$, there are m zeros, and there are no off-particle zeros. The exact ground state of the microscopic Hamiltonian with the Coloumb interaction does not have this simple property. Nevertheless, the ideal Laughlin wavefunction captures all of the essential topological properties of these phases.

The Laughlin wavefunctions are also ground states of certain ideal Hamiltonians that contain interaction potentials that are constructed only from δ-functions and their derivatives. For example, if the interaction potential is of the form

$$V_1(z_1, z_2) = \delta(z_1 - z_2), \tag{5}$$

then the following wavefunction is an exact ground state, with vanishing total potential energy:

$$\Phi_{1/2} = \prod_{i<j}(z_i - z_j)^2. \tag{6}$$

On the other hand, if the interaction potential between two electrons is of the form

$$V_2(z_1, z_2) = v_0\delta(z_1 - z_2) + v_2\partial_{z_1^*}^2\delta(z_1 - z_2)\partial_{z_1}^2, \tag{7}$$

with $v_0 > 0$ and $v_2 > 0$, then the zero-energy ground state will be

$$\Phi_{1/4} = \prod_{i<j}(z_i - z_j)^4. \tag{8}$$

More complicated ground states can be obtained through more complicated interactions. For example, consider the following three-body interaction between electrons:

$$\begin{aligned} V_{\rm Pf} =& \mathcal{S}(v_0\delta(z_1 - z_2)\delta(z_2 - z_3) \\ & - v_1\delta(z_1 - z_2)\partial_{z_3^*}\delta(z_2 - z_3)\partial_{z_3}), \end{aligned} \tag{9}$$

where $\mathcal{S}$ is the total symmetrization operator between z_1, z_2, and z_3. An exact zero-energy ground state of this interaction is the Pfaffian wavefunction at $\nu = 1$:

$$\Phi_{\rm Pf} = \mathcal{A}\left(\frac{1}{z_i - z_j}\right)\prod_{i<j}(z_i - z_j), \tag{10}$$

where $\mathcal{A}$ is the total antisymmetrization operator between $z_1, \cdots, z_N$.

Since the Pfaffian and Laughlin wavefunctions are exact zero-energy ground states of interaction potentials that are constructed only from δ-functions and their derivatives, these wavefunctions can be characterized completely by their pattern of zeros, i.e. by the order of the zeros in the wavefunction as different numbers of particles are taken together. These wave functions have certain ideal properties; for example, the zeros that are bound to a particle lie precisely at the location of that particle, not slightly away from it. In this paper, we will classify ideal FQH wavefunctions, which are wave functions that are exact zero energy ground states of such ideal Hamiltonians.

The existence of these ideal Hamiltonians is crucial. We can write down any arbitrary complex function of N variables, but we cannot know whether it corresponds to a valid topological phase of matter in the limit $N \to \infty$ unless we also know that it is the ground state of some local, gapped Hamiltonian. If this is the case, then it is conceivable that there exists some physical situation in which the low energy effective interactions between electrons yields a phase of matter that is in the same universality class as the ideal wave functions that we constructed. Unfortunately, judging whether a many-particle interacting Hamiltonian is gapped in the thermodynamic limit is in general an intractable problem. So here we limit ourselves to classifying ideal wave functions, which at least we believe can be realized as ground states of local Hamiltonians. Whether the corresponding Hamiltonian can be gapped in the thermodynamic limit is a question that we must attempt to answer along the way.

The ideal quasihole wavefunctions for these states are also zero energy eigenstates of their corresponding ideal Hamiltonians. The true ground state is distinguished by being the unique translationally invariant state with lowest angular momentum. Thus we can also classify topologically distinct quasiholes by their pattern of zeros, i.e. by the order of the zeros in the quasihole wavefunction as different numbers of electrons are taken to the quasihole. While the quasihole wavefunctions are also zero energy states, the quasiparticles will be gapped and will have the same topological properties as the quasiholes. Therefore in the following, when we discuss the pattern of zeros of the quasiparticles, we are referring to the pattern of zeros of the quasihole wave functions.

Not all FQH phases have such ideal wave functions. The hierarchy states and the composite fermion states, for example, do not have ideal wave functions. While these phases do have their own model wave functions, they are not ideal in the sense that they cannot be directly described by their pattern of zeros or written as a correlation function of conformal primary operators in a CFT. Therefore the pattern-of-zeros construction does not directly classify these phases. We will however discuss how they are related to the ideal multilayer FQH wave functions in section IX A.

IV. PATTERN OF ZEROS CHARACTERIZATION

The spirit of the pattern-of-zeros approach is to consider bringing together a_I particles of type I, for $I = 1, \cdots, N_f$, and asking how Φ goes to zero under such a procedure. The order of the zero will be denoted $S_{\vec{a}}$, where $\vec{a} = (a_1, \cdots, a_{N_f})$. In the following we will more precisely define $S_{\vec{a}}$ and discuss some different yet equivalent ways of characterizing the pattern of zeros. This discussion is a straightforward generalization of the discussion in the single-layer case.[10]

A. $S_{\vec{a}}$ characterization

Consider a set of a_I coordinates of each type I, and set $\vec{a} = (a_1, \cdots, a_{N_f})$. Define $S_{\vec{a}}$ as the minimal power of $(\prod_{I=1}^{N_f}\prod_{i=1}^{a_I} z_i^I)$ in the polynomial Φ. This means that if we set

$$\begin{aligned} z_i^I &= \lambda\xi_i^I + z^{(\vec{a})} \qquad i = 1, \cdots, a_I, \quad \forall I, \\ z^{(\vec{a})} &= \frac{\sum_{I=1}^{N_f}\sum_{i=1}^{a_I} z_i^I}{\sum_I a_I}, \qquad \sum_{i,I}\xi_i^I = 0, \end{aligned} \tag{11}$$

and we take $\lambda \to 0$, then

$$\Phi \sim \lambda^{S_{\vec{a}}} P(\{\xi_i^I\}, z^{(\vec{a})}, \{z^I_{a_I+1}, \cdots\}) + O(\lambda^{S_{\vec{a}}+1}), \quad (12)$$

where $P(\{\xi_i^I\}, z^{(\vec{a})}, \{z^I_{a_I+1}, \cdots\})$ is a polynomial in $\{\xi_i^I\}$ and the remaining coordinates $z^{(\vec{a})}$ and $\{z^I_{a_I+i}\}$. We refer to $z^{(\vec{a})}$ as the coordinate of an $\vec{a}$-cluster. We assume that $S_{\vec{a}}$ is independent of the choice of $z^{(\vec{a})}$, which must be the case for translationally invariant wavefunctions. We also assume that $S_{\vec{a}}$ is independent of the choice of $\{\xi_i^I\}$ and that different polynomials $P(\{\xi_i^I\}, z^{(\vec{a})}, \{z^I_{a_I+1}, \cdots\})$ obtained from different choices of ξ_i^I are linearly dependent. This is the assumption of *unique fusion* .

We can immediately deduce some basic properties of $S_{\vec{a}}$. Since Φ has no poles, it is clear that $S_{\vec{a}} \geq 0$. Since Φ must be single-valued under rotating λ in the complex plane by an angle 2π, $S_{\vec{a}}$ must be an integer. Let $S_{\vec{e}_I}$ be the minimal power of z_1^I; that is, $(\vec{e}_I)_J = \delta_{IJ}$. A translationally invariant Φ will have $S_{\vec{e}_I} = 0$, otherwise it will vanish everywhere.

Thus, for a translationally invariant polynomial, $S_{\vec{a}}$ is a nonnegative integer that characterizes the order of zero that results when the size of an $\vec{a}$-cluster goes to zero.

B. Derived Polynomials and the $D_{\vec{a}\vec{b}}$ characterization

In the previous section, we introduced the derived polynomials $P(\{\xi_i^I\}, z^{(\vec{a})}, \{z^I_{a_I+1}, \cdots\})$. As a consequence of the unique fusion condition, these polynomials are actually independent of $\{\xi_i^I\}$. We may consider more general derived polynomials by bringing together other sets of coordinates in P to obtain $\tilde{P}(z^{(\vec{a})}, z^{(\vec{b})}, \cdots)$. Then we may consider bringing together an $\vec{a}$-cluster and a $\vec{b}$-cluster:

$$\begin{aligned}\tilde{P}(z^{(\vec{a})}, z^{(\vec{b})}, \cdots)|_{z^{(\vec{a})} \to z^{(\vec{b})} \equiv z^{(\vec{a}+\vec{b})}} \\ \sim (z^{(\vec{a})} - z^{(\vec{b})})^{D_{\vec{a}\vec{b}}} \tilde{P}'(z^{(\vec{a}+\vec{b})}, z^{(\vec{c})}, \cdots) \\ + O((z^{(\vec{a})} - z^{(\vec{b})})^{D_{\vec{a}\vec{b}}+1}). \end{aligned} \quad (13)$$

Thus, $D_{\vec{a}\vec{b}}$ characterizes the order of the zeros in the derived polynomials as a cluster of $\vec{a}$ electrons are brought close to a cluster of $\vec{b}$ electrons. The unique-fusion condition assumes that the derived polynomials obtained from different ways of fusion are always linearly dependent.

The fact that Φ is a single-valued, symmetric polynomial implies

$$D_{\vec{a}\vec{b}} = D_{\vec{b}\vec{a}} \in \mathbb{Z}, \quad D_{\vec{a}\vec{a}} = \text{even}, \quad D_{\vec{a}\vec{b}} \geq 0. \quad (14)$$

We can deduce a relation between $D_{\vec{a}\vec{b}}$ and $S_{\vec{a}}$ as follows. The order of the zero obtained by creating an $(\vec{a}+\vec{b})$-cluster is $S_{\vec{a}+\vec{b}}$. One way of creating such a cluster is by first creating an $\vec{a}$-cluster, then creating a $\vec{b}$-cluster, and finally bringing together the two clusters to create an $(\vec{a}+\vec{b})$-cluster. The order of zero in this case will be $S_{\vec{a}} + S_{\vec{b}} + D_{\vec{a}\vec{b}}$. Thus $D_{\vec{a}\vec{b}}$ can be obtained from $\{S_{\vec{a}}\}$ through the formula

$$D_{\vec{a}\vec{b}} = S_{\vec{a}+\vec{b}} - S_{\vec{a}} - S_{\vec{b}}. \quad (15)$$

Since $S_{\vec{e}_I} = 0$, where recall $(\vec{e}_I)_J = \delta_{IJ}$ is the unit vector in the I direction, we also have

$$S_{\vec{a}+\vec{e}_I} = S_{\vec{a}} + D_{\vec{a},\vec{e}_I}. \quad (16)$$

From this recursion relation and from the fact that $S_{\vec{e}_I} = 0$, we may obtain $S_{\vec{a}}$ from the sequence $D_{\vec{a}\vec{b}}$. Therefore we may equivalently label the pattern-of-zeros data using $\{S_{\vec{a}}\}$ or $\{D_{\vec{a}\vec{b}}\}$.

C. Characterization by sequence of highest occupied orbitals

The integer $S_{\vec{a}}$ has the following meaning. A polynomial with a_I particles of I^{th} kind (*ie* a_I particles in the I^{th} layer) has a total order of $S_{\vec{a}}$. In other words the total angular momentum of the quantum Hall droplet is $S_{\vec{a}}$ if the droplet has a^I particles in the I^{th} layer. If we remove one particle from the I^{th} layer, then the total angular momentum of the quantum Hall droplet will be reduced to $S_{\vec{a}-\vec{e}_I}$. Thus we can interpret

$$l^I_{\vec{a}} \equiv S_{\vec{a}} - S_{\vec{a}-\vec{e}_I} \quad (17)$$

as the angular momentum of the highest occupied orbital in the I^{th} layer for a quantum Hall droplet with a^J particles in the J^{th} layer. The N_f-dimensional sequence of vectors $\vec{l}_{\vec{a}} = (l^1_{\vec{a}}, ..., l^{N_f}_{\vec{a}})$ will be called the sequence of highest occupied orbitals (HOO).

We see that $l^I_{\vec{a}}$ makes sense only when $a^I > 0$. We will set $l^I_{\vec{a}} = 0$ when it does not make sense. From (17), we also see that there is one-to-one correspondence between the sequence $S_{\vec{a}}$ and $\vec{l}_{\vec{a}}$. Thus we can also use $\vec{l}_{\vec{a}}$ to characterize the pattern of zeros in the wave function.

D. Relation to Angular Momentum on the Sphere

A FQH wave function $\Phi(\{z_i^I\})$ defined on a sphere forms a representation of $SU(2)$. In such a case, z represents the stereographic projection onto the plane of a point on the sphere. A single particle in the lowest Landau level can fill any of the $N_\Phi + 1$ orbitals, so the representation of $SU(2)$ formed in this case is the one with angular momentum $J = N_\Phi/2$. The SU(2) Lie algebra is generated by

$$L^z = z\partial_z - J, \quad L^- = \partial_z, \quad L^+ = -z^2\partial_z + 2Jz. \quad (18)$$

In the multilayer case, then, the angular momentum of a particle of type I will be $J_I = N^I_\Phi/2$ where N^I_Φ is the total number of flux quanta through the sphere seen by the particles in the I^{th} layer. Note that here we allow the

numbers of flux quanta in different layers to be different. The total angular momentum of an $\vec{a}$ cluster in the z direction will be the eigenvalue of the operator

$$L^z_{\vec{a}} = \sum_I (\sum_{i=1}^{a_I} z_i^I \partial_{z_i^I} - J_I). \tag{19}$$

The operator $\sum_I \sum_{i=1}^{a_I} z_i^I \partial_{z_i^I}$ counts the total power of a polynomial. Since the minimum total power of $\prod_I \prod_{i=1}^{a_I} z_i^I$ is $S_{\vec{a}}$, the minimum total angular momentum of an $\vec{a}$-cluster is given by $S_{\vec{a}} - \sum_I a_I J_I$. This means that the $\vec{a}$-cluster carries an angular momentum of

$$J_{\vec{a}} = \vec{a} \cdot \vec{J} - S_{\vec{a}} = \frac{1}{2}\vec{a} \cdot \vec{N}_\Phi - S_{\vec{a}}, \tag{20}$$

where $\vec{J} = (J_1, ..., J_{N_f})$ and $\vec{N}_\Phi = (N_\Phi^1, ..., N_\Phi^{N_f})$. We will use this relation later to construct ideal Hamiltonians and to place conditions on the pattern of zeros for when they can correspond to rotationally invariant wave functions on the sphere.

V. CONSISTENCY CONDITIONS

For the pattern of zeros to describe a valid FQH wavefunction, it must satisfy certain consistency conditions. We already encountered several such conditions above. For instance, we found that $S_{\vec{a}}$ is a nonnegative integer, $D_{\vec{a}\vec{b}} = S_{\vec{a}+\vec{b}} - S_{\vec{a}} - S_{\vec{b}} \geq 0$, and $D_{\vec{a}\vec{a}} = S_{2\vec{a}} - 2S_{\vec{a}}$ is even. In the following we find additional conditions that the pattern of zeros must satisfy.

A. Concave condition

One of the most important conditions on the wave function is simply the condition that the wave function have no poles. This condition is remarkably restrictive on the allowed pattern-of-zeros sequences. Consider a derived polynomial $P(z^{(\vec{a})}, z^{(\vec{b})}, \cdots)$ and fix all coordinates but $z^{(\vec{a})}$, thus viewing it as a complex function $f(z^{(\vec{a})})$. $f(z^{(\vec{a})})$ has zeros at isolated points, but no poles anywhere. Some of the zeros are located at $z^{(\vec{b})}$, $z^{(\vec{c})}$, *etc* . Those zeros are called on-particle zeros. The rest of the zeros are called off-particle zeros.

If we imagine taking $z^{(\vec{a})}$ around $z^{(\vec{b})}$ without enclosing any off-particle zeros, then f will pick up a phase $2\pi D_{\vec{a}\vec{b}}$. Similarly, if we take $z^{(\vec{a})}$ around $z^{(\vec{c})}$ without enclosing any off-particle zeros, then f will pick up a phase $2\pi D_{\vec{a}\vec{c}}$. Now consider taking $z^{(\vec{b})} \to z^{(\vec{c})}$. Under such a process, some nearby off-particle zeros will also be taken to $z^{(\vec{c})}$. Therefore, if we take $z^{(\vec{a})}$ around a contour that encloses both $z^{(\vec{b})}$ and $z^{(\vec{c})}$ in the limit that $z^{(\vec{b})} \to z^{(\vec{c})}$, the complex function f must change by a phase that is greater than or equal to $2\pi(D_{\vec{a}\vec{b}} + D_{\vec{a}\vec{c}})$. The phase can never be less than this amount because that would require the existence of off-particle poles that get taken to $z^{(\vec{c})}$ in order to diminish the strength of the on-particle zeros. By definition, the phase change of f under the above procedure is $2\pi D_{\vec{a},\vec{b}+\vec{c}}$. Therefore, the condition that the wavefunction have no poles leads directly to the following concavity condition on the integers $D_{\vec{a}\vec{b}}$:

$$D_{\vec{a},\vec{b}+\vec{c}} \geq D_{\vec{a}\vec{b}} + D_{\vec{a}\vec{c}}. \tag{21}$$

In cases where all of the zeros are located on the particles and there are no off-particle zeros, the above inequality is saturated. This occurs in the Laughlin states $\Phi = \prod_{i<j}(z_i - z_j)^m$, and their multilayer Abelian generalizations, the Halperin states

$$\Phi = \prod_{I;i<j} (z_i^I - z_j^I)^{K_{II}} \prod_{I<J;i,j} (z_i^I - z_j^J)^{K_{IJ}}. \tag{22}$$

In the following we will rewrite the concave condition as

$$\Delta_3(\vec{a}, \vec{b}, \vec{c}) \geq 0, \tag{23}$$

$$\begin{aligned}\Delta_3(\vec{a}, \vec{b}, \vec{c}) \equiv & D_{\vec{a},\vec{b}+\vec{c}} - D_{\vec{a}\vec{b}} + D_{\vec{a}\vec{c}} \\ = & S_{\vec{a}+\vec{b}+\vec{c}} - S_{\vec{a}+\vec{b}} - S_{\vec{a}+\vec{c}} - S_{\vec{b}+\vec{c}} \\ & + S_{\vec{a}} + S_{\vec{b}} + S_{\vec{c}}.\end{aligned} \tag{24}$$

B. Cluster Condition

The cluster condition is a way to associate some kind of grading to the polynomials that is physically meaningful. Let $\{\vec{n}_I\}$ for $I = 1, \cdots, N_f$ be a set of vectors that generate an N_f-dimensional lattice, where N_f is, as before, the number of flavors of particles (or the number of layers). The cluster condition states that the concave condition is saturated, *i.e.*

$$D_{\vec{a},\vec{b}+\vec{c}} = D_{\vec{a}\vec{b}} + D_{\vec{a}\vec{c}}, \tag{25}$$

if either $\vec{a}$, $\vec{b}$, or $\vec{c}$ lie on the lattice generated by $\{\vec{n}_I\}$. That is, if either $\vec{a}$, $\vec{b}$, or $\vec{c}$ can be written as a linear combination with integer coefficients of the vectors $\{\vec{n}_I\}$. This means that a derived polynomial containing a $\vec{k} = \sum_I k_I \vec{n}_I$ cluster is non-zero unless $z^{(\vec{k})}$ coincides with the coordinates of another cluster; viewed as a function of the single variable $z^{(\vec{k})}$, it has no off-particle zeros. A consequence of this is that if all of the particles are fused to form $\vec{n}$-clusters, then the resulting derived polynomial has the Laughlin-Halperin form (see (22)) and there are no off-particle zeros.

The single-layer Read-Rezayi Z_n parafermion wave functions satisfy an n-cluster condition and they are exact ground states of Hamiltonians with $n+1$-body interactions. For a fixed filling fraction, as n increases, the number of topologically distinct quasiparticles, the ground state degeneracy on higher genus surfaces and the

complexity of interactions necessary to realize the state all increase. This suggests that the energy gap typically decreases with increasing n. Wave functions that do not obey a cluster condition can be thought of as having infinite n and are not expected to correspond to gapped phases. This intuition also comes from the CFT approach to FQH wave functions; infinite n corresponds to an irrational conformal field theory, which does not yield a finite number of quasiparticles and a finite ground state degeneracy on the torus. In the multilayer case, we may use the volume of the unit cell spanned by $\{\vec{n}_I\}$ as one way to measure the complexity of a given FQH state.

The cluster condition is extremely powerful and simplifying because it allows us to determine the entire pattern-of-zeros sequence from knowledge of a "small" number of them. To see how this works, first observe using (25) that

$$D_{\vec{n}_J,\vec{a}} = \sum_I a_I D_{\vec{n}_J,\vec{e}_I} \equiv \sum_I m_{JI} a_I, \tag{26}$$

where we have defined the matrix $m_{JI} \equiv D_{\vec{n}_J,\vec{e}_I}$. So for any vector $\vec{k} = \sum_I k_I \vec{n}_I$, where k_I is an integer and $\sum_I k_I (\vec{n_I})_J \geq 0$, we have:

$$D_{\vec{k},\vec{a}} = \sum_{IJ} k_I m_{IJ} \vec{a}_J. \tag{27}$$

The above equations imply

$$\begin{aligned} D_{\vec{n}_I,\vec{n}_J} &= \sum_A n_{JA} D_{\vec{n}_I,\vec{e}_A} = \sum_A n_{JA} m_{IA} = (nm^T)_{JI} \\ &= \sum_A n_{IA} D_{\vec{e}_A,\vec{n}_J} = \sum_I n_{IA} m_{JA} = (mn^T)_{JI}, \end{aligned} \tag{28}$$

where we have also defined the matrix $n_{IJ} = (\vec{n}_I)_J$.

In terms of the sequence $\{S_{\vec{a}}\}$, this implies that for $\vec{k} = \sum_I k_I \vec{n}_I$, where k_I is an integer,

$$\begin{aligned} S_{\vec{a}+\vec{k}} = S_{\vec{a}} &+ \sum_I k_I S_{\vec{n}_I} + \sum_{IJ} k_I m_{IJ} a_J \\ &+ \frac{1}{2} \sum_{IJ} (nm^T)_{JI} (k_I k_J - \delta_{IJ} k_I). \end{aligned} \tag{29}$$

Therefore, all of the integers $S_{\vec{a}}$ are specified by the points $\vec{a}$ within the unit cell spanned by $\{\vec{n}_I\}$.

In terms of the HOO squence $\vec{l_{\vec{a}}}$, we have:

$$\begin{aligned} l^I_{\vec{a}+\vec{k}} &= S_{\vec{a}+\vec{k}} - S_{\vec{a}+\vec{k}-\vec{e}_I} \\ &= S_{\vec{a}} - S_{\vec{a}-\vec{e}_I} + \sum_A k_A m_{AI} \\ &= l^I_{\vec{a}} + \sum_A k_A m_{AI}. \end{aligned} \tag{30}$$

Finally, note that since $D_{\vec{a},\vec{a}}$ is an even integer, we have:

$$\begin{aligned} \text{even } &= D_{\vec{n}_J \vec{n}_J} = \sum_I n_{JI} D_{\vec{n}_J,\vec{e}_I} = \sum_I n_{JI} m_{JI} \\ &= (nm^T)_{JJ}. \end{aligned} \tag{31}$$

C. Equal Area Layers

The density profiles of the single-particle states in the lowest Landau level, $z^m e^{-|z|^2/4l_B^2}$, are in the shape of a ring, with a peak at a radius $r_m = \sqrt{2m} l_B$, where l_B is the magnetic length. Such a wave function has an angular momentum m. When many of these orbitals are filled by particles, the total wavefunction will describe a uniform, rotationally symmetric state that goes to zero at a radius $r_{max} = \sqrt{2m_{max}} l_B$, where m_{max} refers to the filled orbital with maximum angular momentum. Therefore, a given quantum Hall wavefunction will describe a QH droplet of area $4\pi m_{max} l_B^2$, and m_{max} is given by the maximum power of z_1 (or z_i for any other fixed i). m_{max} is also equal to the number of flux quanta, N_Φ.

An important constraint on the multilayer quantum Hall wavefunctions is that they must describe systems in which each layer occupies the same area, up to small corrections. The requirement that each layer occupies exactly the same area amounts to the requirement that each layer has exactly the same number of flux quanta, $N_\Phi^I = N_\Phi^J \equiv N_\Phi$. However, it is reasonable to include states in which different layers occupy equal areas only up to $O(N_e^0)$ corrections, where N_e is the number of electrons.

Such a requirement of approximately equal area layers is summarized in the following equation:

$$\lim_{N_e \to \infty} \frac{N_\Phi^I}{N_\Phi^1} = 1. \tag{32}$$

We wish to see how this condition translates into a condition on the pattern of zeros. The conditions are slightly different depending on whether we ultimately want to characterize gapped FQH phases of fermions or bosons. If we are interested in fermionic phases, we require that $\Psi(\{z_i^I\}) = \prod_{I;i<j} (z_i^I - z_j^I) \Phi(\{z_i^J\})$ be a valid FQH wave function of fermions, which does not require that $\Phi(\{z_i^I\})$ be a valid FQH wave function of bosons. In what follows we will explicitly analyze the bosonic case, where we require $\Phi(\{z_i^I\})$ to be a valid multilayer FQH wave function of bosons.

N_Φ^I is equal to the maximal power of z_1^I; for the boson wavefunction Φ, this is given by

$$N_\Phi^I = S_{\vec{N}} - S_{\vec{N}-\vec{e}_I}, \tag{33}$$

where recall N_I is the number of particles of type I. Using the cluster condition, we find

$$N_\Phi^J = \sum_I N_I (n^{-1}m)_{IJ} + S_{\vec{n}_J} - S_{\vec{n}_J - \vec{e}_J} - m_{JJ}, \tag{34}$$

where we have set $\vec{N} = \sum_I k_I \vec{n}_I$. Requiring (32), we obtain the following condition on the pattern of zeros:

$$\sum_I (m^{-1}n)_{IJ} \geq 0. \tag{35}$$

This can be seen most easily by ignoring the $O(N_e^0)$ terms in (34), taking $N_\Phi^I/N_\Phi \to 1$, and inverting the result to obtain $N_I \sim N_\Phi \sum_I (m^{-1}n)_{IJ}$, which must be nonnegative. From this analysis, we learn that if $n^{-1}m$ is not invertible, then the pattern of zeros cannot fix the ratio of particles N_I/N_1 in the different layers. Therefore the corresponding FQH state has a gapless mode corresponding to the relative density fluctuations between the different layers.

As a simple example of this analysis, consider the $(1,1,1)$ Halperin bilayer state, which is known to have a gapless density mode and for which $n^{-1}m = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, which is not invertible. A macroscopic number of particles can freely go from one layer to the other without changing the area of the quantum Hall droplets, signalling the existence of a gapless relative density mode.

Inverting (34) yields

$$\sum_I (m^{-1}n)_{IJ}(N_\Phi^I - S_{\vec{n}_I} + S_{\vec{n}_I - \vec{e}_I} + m_{II}) = N_J, \quad (36)$$

from which we can read off the filling fraction in each layer:

$$\nu_I = \sum_J (m^{-1}n)_{IJ}. \quad (37)$$

The total filling fraction is the sum of the filling fraction of each layer: $\nu = \sum_I \nu_I$.

For fermions, (33) is modified to

$$N_\Phi^I = S_{\vec{N}} - S_{\vec{N}-\vec{e}_I} + N_I - 1, \quad (38)$$

due to the extra factor $\prod_{I;i<j}(z_i^I - z_j^I)$ in $\Psi(\{z_i^I\})$. Note that $\{S_{\vec{a}}\}$ still describes the pattern of zeros of the symmetric polynomial Φ. The result for fermions is therefore

$$\nu_I = \sum_J (\mathbb{I} + n^{-1}m)^{-1}_{IJ} \geq 0, \quad (39)$$

where $\mathbb{I}$ is the $N_f \times N_f$ identity matrix. If $(\mathbb{I} + n^{-1}m)$ is not invertible in the fermionic case, then there are gapless relative density modes, which is why the filling fraction in each layer becomes undefined.

D. Shift and Rotational Invariance on the Sphere

Consider a multilayer quantum state with N_I particles in the I^{th} layer. We want to put the quantum state on a sphere with N_Φ^I flux quanta in the I^{th} layer. We would like to know for which set of N_Φ^I can the quantum Hall state completely fill the sphere? Naively, one may expect N_Φ^I and N_I are related by the filling fraction in each layer $N^I/\nu_I = N_\Phi^I$. However the precise relation between the number of flux quanta and the number of electrons includes a shift,

$$\nu^{-1} \sum_I \nu_I N_\Phi^I = \nu^{-1} N_e - \mathcal{S}, \quad (40)$$

where $\mathcal{S}$ is of order 1 in the large N_e limit (see eqn. (36)).

More precisely, completely filling the sphere means that the quantum Hall state is rotationally invariant with zero total angular momentum. Using (20), we find that, for a bosonic FQH state characterized by $S_{\vec{a}}$ with N_I particles and N_ϕ^I flux quanta in the I^{th} layer, the maximum total angular momentum is given by

$$J_{\vec{N}} = \frac{1}{2}\vec{N} \cdot \vec{N}_\Phi - S_{\vec{N}} \quad (41)$$

where $\vec{N} = (N_1, ..., N_{N_f})$ and $\vec{N}_\Phi = (N_\Phi^1, ..., N_\Phi^{N_f})$. For a fermionic FQH state characterized by $S_{\vec{a}}$ with N_I particles and N_ϕ^I flux quanta in the I^{th} layer, the maximum total angular momentum is given by

$$J_{\vec{N}} = \frac{1}{2}\vec{N} \cdot \vec{N}_\Phi - S_{\vec{N}} - \frac{1}{2}\sum_I N_I(N_I - 1). \quad (42)$$

In the above, $\vec{N}_\Phi$ must satisfy

$$\begin{array}{ll} N_\Phi^I \geq S_{\vec{N}} - S_{\vec{N}-\vec{e}_I} & \text{for bosons} \\ N_\Phi^I \geq S_{\vec{N}} - S_{\vec{N}-\vec{e}_I} + N_I - 1 & \text{for fermions} \end{array} \quad (43)$$

in order for the wave function to fit into each layer.

Completely filling the sphere requires that

$$\begin{array}{ll} N_\Phi^I = S_{\vec{N}} - S_{\vec{N}-\vec{e}_I} & \text{for bosons} \\ N_\Phi^I = S_{\vec{N}} - S_{\vec{N}-\vec{e}_I} + N_I - 1 & \text{for fermions} \end{array} \quad (44)$$

and $J_{\vec{N}} = 0$. We see that $\vec{N}$ and $\vec{N}_\Phi$ must satisfy

$$\vec{N} \cdot \vec{N}_\Phi = \begin{cases} 2S_{\vec{N}} & \text{for bosons} \\ 2S_{\vec{N}} + \sum_I N_I(N_I - 1) & \text{for fermions} \end{cases} \quad (45)$$

which implies (for both bosons and fermions)

$$\sum_I N_I(S_{\vec{N}} - S_{\vec{N}-\vec{e}_I}) = 2S_{\vec{N}}. \quad (46)$$

If a given $\vec{N}$ does not satisfy (46), then the corresponding quantum Hall state (with N_I particles on the I^{th} layer) cannot completely fill the sphere. For $\vec{N}$ that satisfies (46), the corresponding quantum Hall state can completely fill the sphere and has zero total angular momentum if $\vec{N}_\Phi$ is given by (44). (46) can generally be satisfied only if $\vec{N}$ lies on the lattice spanned by $\{\vec{n}_I\}$.

We would like to remark that it is easy to have different numbers of flux quanta on different layers in numerical calculations. The pattern of $(\vec{N}, \vec{N}_\Phi)$ where the quantum Hall state has zero total angular momentum on the sphere can be used as a fingerprint to identify different quantum Hall states through numerical calculations (for examples, see Tables IV and V).

E. Additional constraints: $\Delta_3 =$ even

The analysis of the single-layer case in Ref. 10 has suggested an additional condition:

$$\Delta_3(a,b,c) = \text{even}. \tag{47}$$

There, it was found that allowing $\Delta_3(a,b,c) =$ odd allows for certain pattern-of-zeros sequences that either do not correspond to single-valued wavefunctions (such as the square root of the Pfaffian) or could not correspond to translationally invariant wavefunctions. It was suggested that one way to rule out such possibilities is to impose (47). How should this condition be generalized to the multilayer situation?

One natural generalization is to impose $\Delta_3(\vec{a},\vec{b},\vec{c}) =$ even for all $\vec{a}$, $\vec{b}$, and $\vec{c}$. However, we find that this condition is too restrictive. It rules out certain known FQH wavefunctions, such as the $su(3)_2/u(1)^2$ non-Abelian spin singlet states.[17,18] The need to relax this condition while still having it remain compatible with the single-layer situation suggests that we should impose (47) only for choices of $\vec{a}$, $\vec{b}$, $\vec{c}$ that are collinear through the origin.

While it was found that allowing $\Delta_3(a,b,c) =$ odd allows for pattern-of-zeros sequences that do not seem to correspond to valid translationally invariant, single-valued wavefunctions, there are known cases of CFTs with $\Delta_3(a,b,c) =$ odd that do seem to yield translationally invariant, single-valued wavefunctions. One such example is the so-called Gaffnian wavefunction, which has $\Delta_3(1,1,1) =$ odd and which can be constructed using the minimal model CFT $\mathcal{M}(5,3)$.[19] This CFT however is non-unitary. It has been suggested that FQH wavefunctions constructed using non-unitary CFTs correspond to gapless phases;[20] whether this is always necessarily the case is currently an important open question in FQH theory.

These considerations suggest that in order to restrict ourselves to pattern-of-zeros sequences that have a corresponding unitary CFT, we should impose $\Delta_3(\vec{a},\vec{b},\vec{c}) =$ even for those $\vec{a}$, $\vec{b}$, and $\vec{c}$ that are collinear through the origin. In our search for pattern-of-zeros solutions, we will impose this condition and analyze the resulting states. The precise connection, if any, between this condition and valid FQH wavefunctions that correspond to unitary CFTs remains to be clarified.

VI. IDEAL HAMILTONIANS

Given a pattern-of-zeros sequence, it is important to be able to construct a local, gapped Hamiltonian whose ground state wave function has the given pattern of zeros. If this is possible, then we know that the corresponding pattern-of-zeros sequence describes a topological phase of matter. Whether this particular phase is realized in an experiment then depends on the particular types of low energy effective interactions between the electrons in the unfilled Landau levels.

We can go about constructing such a Hamiltonian by noticing that on a sphere, the integers $S_{\vec{a}}$ are directly related to the angular momentum of the $\vec{a}$-cluster. For an electron system on a sphere with N_ϕ^I flux quanta for the Ith layer, an electron of type I will carry an angular momentum $J^I = N_\Phi^I/2$. For an $\vec{a}$-cluster, the maximum angular momentum is therefore $\vec{a}\cdot\vec{J}$. However, for a polynomial $\Phi(\{z_i^I\})$ described by a pattern of zeros $\{S_{\vec{a}}\}$, the maximum allowed angular momentum of the $\vec{a}$ cluster is only $J_{\vec{a}} = \vec{a}\cdot\vec{J} - S_{\vec{a}}$. The pattern of zeros forbids the appearance of angular momentum $\vec{a}\cdot\vec{J} - S_{\vec{a}} + 1, \vec{a}\cdot\vec{J} - S_{\vec{a}} + 2, \cdots, \vec{a}\cdot\vec{J}$ for any $\vec{a}$-clusters in $\Phi(\{z_i^I\})$.

Such a condition can be easily enforced by writing the Hamiltonian as a sum of projection operators, $P_S^{(\vec{a})}$. Let $P_S^{(\vec{a})}$ be a projection operator that acts on the $\vec{a}$-cluster Hilbert space. $P_S^{(\vec{a})}$ projects onto the subspace of $\vec{a}$-clusters that have total angular momenta greater than $\vec{a}\cdot\vec{J} - S$. Now consider the Hamiltonian

$$H_{\{S_{\vec{a}}\}} = \sum_{\vec{a}} \sum_{\vec{a}\text{-clusters}} P_{S_{\vec{a}}}^{(\vec{a})}, \tag{48}$$

where $\sum_{\vec{a}\text{-clusters}}$ sums over all of the $\vec{a}$-clusters for a fixed $\vec{a}$. The wave function described by $\{S_{\vec{a}}\}$ will clearly be a zero-energy ground state of the above $H_{\{S_{\vec{a}}\}}$. In many cases, there is only one unique ground state wave function with minimal total angular momentum, but in general there can be many independent polynomials with the same pattern of zeros. In such a situation, the Hamiltonian would need to be modified further to select for a particular polynomial with the given pattern of zeros.

In order for the above Hamiltonian to be local, $\sum_{\vec{a}}$ must be limited to a small, finite number of $\vec{a}$-clusters. But as a result, we cannot guarantee in general that the ground state wave functions will all be described by the sequence $\{S_{\vec{a}}\}$ for every $\vec{a}$, or even that they will obey the cluster condition. In many of the known cases, such as the Laughlin, Moore-Read, and Read-Rezayi wave functions, the sum over $\vec{a}$-clusters can indeed be terminated after the first few clusters while still yielding a unique zero energy ground state wave function with minimal angular momentum which is described by $\{S_{\vec{a}}\}$.

The above construction for $H_{\{S_{\vec{a}}\}}$ should therefore be viewed as a starting point for constructing an ideal Hamiltonian that is local, gapped, and whose unique minimal angular momentum ground state wave function is described by $\{S_{\vec{a}}\}$. In some of the simplest cases, we know that this construction suffices.

VII. SUMMARY: PATTERN-OF-ZEROS DATA AND CONDITIONS

We have found that the polynomials $\Phi(\{z_i^I\})$, $I = 1, \cdots, N_f$, that may correspond to stable FQH states are

described by the following data:

$$n, \quad m, \quad \{S_{\vec{a}}\}, \tag{49}$$

where n and m are $N_f \times N_f$ matrices with integer entries that satisfy

$$\begin{aligned} m_{IJ} \geq 0, \qquad n_{IJ} \geq 0, \qquad \det n \neq 0 \\ mn^T = nm^T, \qquad (mn^T)_{II} = \text{ even}. \end{aligned} \tag{50}$$

The above implies that $n^{-1}m$ is a symmetric matrix. Furthermore, for the pattern of zeros to fix the relative densities of particles in each layer, we have

$$\begin{aligned} (n^{-1}m)_{IJ} \quad &\text{is invertible (for bosons)} \\ (n^{-1}m)_{IJ} + \delta_{IJ} \quad &\text{is invertible (for fermions)}. \end{aligned} \tag{51}$$

Otherwise, there are gapless relative density fluctuations. We also have

$$\begin{aligned} &\nu_I \geq 0, \\ &\nu_I = \begin{cases} \sum_J (m^{-1}n)_{IJ} & \text{for bosons} \\ \sum_J (\mathbb{I} + n^{-1}m)^{-1}_{IJ} & \text{for fermions} \end{cases} \end{aligned} \tag{52}$$

The total filling fraction is $\nu = \sum_I \nu_I$.

Note that the $\{S_{\vec{a}}\}$ need to be specified only for values of $\vec{a}$ that are contained in the unit cell spanned by $\{\vec{n}_I\}$, where $\vec{n}_I$ corresponds to the I^{th} row of the matrix n: $(\vec{n}_I)_J = n_{IJ}$.

Using the cluster condition, we can determine $S_{\vec{a}}$ for all $\vec{a}$ simply from m, the fact that the $S_{\vec{e}_I} = 0$, and from the values of $S_{\vec{b}}$ for which $\vec{b}$ lies in the unit cell spanned by $\{\vec{n}_I\}$:

$$\begin{aligned} S_{\vec{a}+\vec{k}} = S_{\vec{a}} + \sum_I k_I S_{\vec{n}_I} + \sum_{IJ} k_I m_{IJ} \vec{a}_J \\ + \frac{1}{2} \sum_{IJ} (nm^T)_{IJ} (k_I k_J - \delta_{IJ} k_I), \end{aligned} \tag{53}$$

where $\vec{k} = \sum_I k_I \vec{n}_I$. $S_{\vec{a}}$ must satisfy:

$$\begin{aligned} \Delta_2(\vec{a}, \vec{a}) = \text{ even}, \\ \Delta_2(\vec{a}, \vec{b}) \geq 0, \quad \Delta_3(\vec{a}, \vec{b}, \vec{c}) \geq 0, \end{aligned} \tag{54}$$

where

$$\Delta_2(\vec{a}, \vec{b}) \equiv S_{\vec{a}+\vec{b}} - S_{\vec{a}} - S_{\vec{b}},$$

$$\begin{aligned} \Delta_3(\vec{a}, \vec{b}, \vec{c}) \equiv & S_{\vec{a}+\vec{b}+\vec{c}} - S_{\vec{a}+\vec{b}} - S_{\vec{a}+\vec{c}} - S_{\vec{b}+\vec{c}} \\ & + S_{\vec{a}} + S_{\vec{b}} + S_{\vec{c}}. \end{aligned} \tag{55}$$

Finally we impose

$$\Delta_3(\vec{a}, \vec{b}, \vec{c}) = \text{ even} \tag{56}$$

for those $\vec{a}$, $\vec{b}$, and $\vec{c}$ that are collinear through the origin.

A. Product of Symmetric Polynomials and Primitive Solutions

Consider two polynomials of $\vec{n}$-cluster form, Φ and Φ', and consider their product: $\tilde{\Phi} = \Phi\Phi'$. The pattern of zeros of $\tilde{\Phi}$ is the sum of the pattern of zeros of Φ and Φ':

$$\tilde{D}_{\vec{\alpha}\vec{\beta}} = D_{\vec{\alpha}\vec{\beta}} + D'_{\vec{\alpha}\vec{\beta}}. \tag{57}$$

Similarly, the data in terms of m and $S_{\vec{\alpha}}$ are also additive. Note that all of the conditions that we impose on the pattern-of-zeros are linear. Thus the pattern of zeros of $\tilde{\Phi}$ is also valid. However, notice that the condition for filling the sphere is not linear. There may be two FQH wavefunctions Φ and Φ' that can fill the sphere but whose product $\tilde{\Phi}$ cannot fill the sphere.

Thus, we can divide the pattern-of-zeros solutions into primitive and non-primitive solutions. Primitive solutions are solutions that cannot be written as a sum of two other solutions.

VIII. RELATION TO CONFORMAL FIELD THEORY

The pattern-of-zeros approach is closely related to the conformal field theory approach to constructing FQH wavefunctions. In the CFT approach, the symmetric polynomial $\Phi(\{z_i^I\})$ that describes a multilayer FQH state can be written as a correlation function of a set of electron operators $V_{e;I}$ in a CFT:[21]

$$\Phi(\{z_i^I\}) = \lim_{z_\infty \to \infty} z_\infty^{2h_{\vec{N}}} \langle V(z_\infty) \prod_{i,I} V_{e;I}(z_i^I) \rangle. \tag{58}$$

The operators $V_{e;I}$ are written in the form:

$$V_{e;I}(z) = \psi_{\vec{e}_I}(z) e^{i \sum_J M_{IJ} \phi_J(z)}, \tag{59}$$

where $e^{i \sum_J M_{IJ} \phi_J(z)}$ is a vertex operator in a $U(1)^{N_f}$ CFT. It has scaling dimension $\sum_J M_{IJ}^2/2$. $\psi_{\vec{e}_I}$ is a simple current operator; that is, it satisfies the following fusion relation:

$$\psi_{\vec{a}} \psi_{\vec{b}} = \psi_{\vec{a}+\vec{b}}. \tag{60}$$

This Abelian fusion rule is the CFT version of the unique-fusion condition. The cluster condition implies that ψ satisfies

$$\psi_{\vec{n}_I} \sim 1, \tag{61}$$

where $(\vec{n}_I)_J = n_{IJ}$. An $\vec{a}$-cluster of electrons will be described by the operator

$$V_{\vec{a}} = \prod_I V_{e;\vec{e}_I}^{a_I} = \psi_{\vec{a}} e^{i \sum_{IJ} a_I M_{IJ} \phi_J(z)}. \tag{62}$$

Thus we see that the cluster condition implies that an $\vec{n}_I$ cluster is described by a vertex operator

$e^{i\sum_{JK} n_{IJ} M_{JK} \phi_K(z)}$. If all of the particles are grouped into n-clusters, then the corresponding derived polynomial will be just a correlation function of vertex operators in a $U(1)^{N_f}$ theory, which will have no off-particle zeros and will be of the Laughlin-Halperin form.

Let us denote the scaling dimension of the operator $V_{\vec{a}}$ as

$$h_{\vec{a}} = h^{\rm sc}_{\vec{a}} + h^{\rm ga}_{\vec{a}}, \tag{63}$$

where $h^{\rm sc}_{\vec{a}}$ is the scaling dimension of the simple current $\psi_{\vec{a}}$ and $h^{\rm ga}_{\vec{a}}$ is the scaling dimension of the vertex operator. Notice that since $\psi_{\vec{n}_I} \sim 1$, the simple current scaling dimensions satisfy $h^{\rm sc}_{\vec{a}+\vec{n}_I} = h^{\rm sc}_{\vec{a}}$. The scaling dimension of the Gaussian part is given by

$$h^{\rm ga}_{\vec{a}} = \frac{1}{2} a_I (MM^T)_{IJ} a_J. \tag{64}$$

The pattern of zeros are related to the scaling dimensions through the relation

$$\begin{aligned} D_{\vec{a},\vec{b}} &= h_{\vec{a}+\vec{b}} - h_{\vec{a}} - h_{\vec{b}} \\ &= S_{\vec{a}+\vec{b}} - S_{\vec{a}} - S_{\vec{b}}. \end{aligned} \tag{65}$$

This allows us to obtain the scaling dimensions from the pattern of zeros. Using the cluster condition, some algebra shows that $MM^T = n^{-1}m$ and so (64) becomes

$$h^{\rm ga}_{\vec{a}} = \frac{\vec{a}^T n^{-1} m \vec{a}}{2}. \tag{66}$$

The scaling dimensions of the simple-current part can also be determined from the pattern of zeros by using the fact that $h_{\vec{n}_I} = \frac{(nm^T)_{II}}{2}$, $h_0 = 0$, and applying (65) iteratively. This yields:

$$\frac{(nm^T)_{II}}{2} = \sum_A n_{IA} h_{\vec{e}_A} + S_{\vec{n}_I}. \tag{67}$$

Multiplying both sides by n^{-1} gives

$$h_{\vec{e}_A} = h^{sc}_{\vec{e}_A} + \frac{\vec{e}_A^T n^{-1} m \vec{e}_A}{2} = n^{-1}_{AI}\left(\frac{(nm^T)_{II}}{2} - S_{\vec{n}_I}\right). \tag{68}$$

In a similar manner, one can obtain

$$\begin{aligned} h_{\vec{a}} &= \sum_I a_I (n^{-1})_{IJ} \left(\frac{(nm^T)_{JJ}}{2} - S_{\vec{n}_J}\right) + S_{\vec{a}}. \\ &= h^{sc}_{\vec{a}} + \frac{\vec{a}^T n^{-1} m \vec{a}}{2}, \end{aligned} \tag{69}$$

which determines $h^{\rm sc}_{\vec{a}}$ in terms of the pattern-of-zeros.

Note that the correlation function of the Gaussian part is, leaving the background charge implicit,

$$\begin{aligned} &\langle \prod_{I;i} e^{\sum_J M_{IJ}\phi_J(z^I_i)} \rangle \\ &\quad = \prod_{I;i<j} (z^I_i - z^I_j)^{(MM^T)_{II}} \prod_{I,J;i,j} (z^I_i - z^J_j)^{(MM^T)_{IJ}} \\ &\quad = \prod_{I;i<j} (z^I_i - z^I_j)^{(n^{-1}m)_{II}} \prod_{I<J;i,j} (z^I_i - z^J_j)^{(n^{-1}m)_{IJ}}. \end{aligned} \tag{70}$$

Thus the FQH wave function is of the form

$$\Phi(\{z^I_i\}) = \Phi_{\rm sc}(\{z^I_i\}) \Phi_{\rm ga}(\{z^I_i\}), \tag{71}$$

where

$$\begin{aligned} &\Phi_{\rm ga}(\{z^I_i\}) = \\ &\quad \prod_{I;i<j} (z^I_i - z^I_j)^{(n^{-1}m)_{II}} \prod_{I<J;i,j} (z^I_i - z^J_j)^{(n^{-1}m)_{IJ}}. \end{aligned} \tag{72}$$

Φ_{sc} arises from the correlation function of the simple current sector and is the "non-Abelian part" of the wavefunction.

In this manner, each pattern-of-zeros solution corresponds to the current algebra of a rational CFT. The connection between the pattern of zeros approach and the CFT approach can be thought of in the following way. The pattern of zeros describes the essential properties of the CFT that yield valid FQH wavefunctions. So in order to classify ideal FQH wavefunctions, one can bypass the CFT altogether and go directly to the heart of the matter: characterizing the allowed pattern-of-zeros solutions. Furthermore, since each pattern-of-zeros solution corrresponds to a CFT, the pattern-of-zeros classification can be viewed as a classification of the allowed CFTs that can be used to construct FQH wave functions.

In this formulation, the pattern of zeros classifies all those ideal FQH wavefunctions that can be formulated as a correlation function of *conformal primary* fields. There are also many FQH wavefunctions, such as the hierarchy states and the Jain series, that cannot be written in this way. These wavefunctions are outside of the pattern-of-zeros classification. However they may be closely related to the pattern-of-zeros construction. We comment on this connection elsewhere.

A. Alternate Labeling

Using (69), we can derive a formula for $S_{\vec{a}}$ in terms of $h^{sc}_{\vec{a}}$, n, and m:

$$S_{\vec{a}} = h_{\vec{a}} - \sum_I a_I h_{\vec{e}_I}. \tag{73}$$

Thus there is a one-to-one correspondence between the simple-current scaling dimensions and the sequence $\{S_{\vec{a}}\}$. This means that there is yet another way to label the pattern of zeros. Earlier, we found that one convenient labeling of the pattern of zeros is with an $N_f \times N_f$ matrix n, an $N_f \times N_f$ matrix m, and the value of the non-negative integers $S_{\vec{a}}$ for $\vec{a}$ lying inside the unit cell spanned by the rows of n. An alternative, equivalent labeling of the pattern of zeros is by specifying the following data:

$$n, \quad m, \quad \{h^{sc}_{\vec{a}}\}, \tag{74}$$

for $\vec{a}$ lying inside the unit cell spanned by n. This labeling is convenient because it makes close contact with the corresponding CFT description; if $h^{sc}_{\vec{a}} \neq 0$ for some $\vec{a}$, then

the CFT has a non-trivial simple-current structure and therefore generally also has some form of non-Abelian statistics.

B. Relevant CFTs for multilayer FQH states: $\hat{g}_k/u(1)^r$ parafermions

In the single-layer case, many of the pattern-of-zeros solutions were found to be closely related to the Z_k parafermion CFTs of Zamalodchikov and Fateev.[22] What are the relevant CFTs to expect in the multilayer case? The answer is that some of the corresponding CFTs in the multilayer case will be closely related to the $\hat{g}_k/u(1)^r$ parafermion CFTs that were constructed by Gepner, where $\hat{g}_k$ is a simple affine Lie algebra at level k and r is the rank of the Lie algebra g.[23] The case $g = su(2)$ is equivalent to the Z_k parafermion CFTs of Zamalodchikov and Fateev.

The simple-current algebra of the $\hat{g}_k/u(1)^r$ parafermion CFT has the following structure. For every element $\vec{\alpha}$ of the r-dimensional root lattice of g, associate a simple current operator $\psi_{\vec{\alpha}}$. The simple-currents will have the fusion rules

$$\psi_{\vec{\alpha}}\psi_{\vec{\beta}} = \psi_{\vec{\alpha}+\vec{\beta}}. \tag{75}$$

Furthermore, $\psi_{\vec{\alpha}} = \psi_{\vec{\beta}}$ if $\vec{\alpha} - \vec{\beta}$ is an element of k times the long root lattice of g. The scaling dimension of $\psi_{\vec{\alpha}}$ is given by

$$h^{sc}_{\vec{\alpha}} = -\frac{\vec{\alpha}^2}{2k} + n(\vec{\alpha}), \tag{76}$$

where $n(\vec{\alpha})$ is an integer equal to the minimum number of roots from which $\vec{\alpha}$ is composed. The inner product $\vec{\alpha}^2 = (\vec{\alpha}, \vec{\alpha})$ is defined with respect to the quadratic form matrix of g.

If we are considering quantum Hall states with N_f layers, then we would expect to see the appearance of these parafermion CFTs with rank $r \geq N_f$. Therefore in the bilayer case, one class of states that we expect to see should be related to $\hat{g}_k/u(1)^2$ parafermion CFTS where g is a simple Lie algebra of rank 2. There are only three simple Lie algebras of rank 2: $su(3)$, $so(5)$, and G_2. Of these, only $su(3)$ is simply laced, so the long root lattice is the same as the root lattice. This means that for the pattern-of-zeros solutions that correspond to $n = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$, we expect to see solutions that correspond to $su(3)_k/u(1)^2$ CFTs. In Appendix B we will describe the $su(3)_2/u(1)^2$ CFT in more detail.

The parafermion CFTs for $g = so(5)$ and $g = G_2$, on the other hand, are more complicated because the long root lattice is different from the root lattice. For example, $so(5)_k/u(1)^2$ CFTs will generically be relevant for $n = \begin{pmatrix} 2k & 0 \\ 0 & 2k \end{pmatrix}$ while $(G_2)_k/u(1)^2$ CFTs will generically be relevant for $n = \begin{pmatrix} 3k & 0 \\ 0 & 3k \end{pmatrix}$.

IX. EXAMPLES OF PATTERN OF ZEROS SOLUTIONS

In this section we examine explicitly several simple bilayer pattern of zeros solutions. We first fix the cluster structure n to have a simple form. Then we try to find all the solutions $\{S_{\vec{a}}\}$ that satisfy the conditions listed in section VII.

The simplest non-Abelian states can be obtained from the simplest cluster structures $n = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ and $n = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. These are the simplest non-Abelian generalizations of the bilayer Abelian Laughlin-Halperin states.

Note that by definition the ordering of the rows in n is arbitrary; we choose it so that $n_{II} \neq 0$. Interchanging the layers yields the same physical system but corresponds to interchanging n_{11} with n_{22} and n_{12} with n_{21}, so two matrices n and n' that are related by such an interchange are regarded as equivalent.

To list the solutions for those simplest cases, we may use some known CFTs to construct the appropriate simple-current algebra that corresponds to the pattern-of-zeros solutions. Using this known CFT, we can then write the wavefunction explicitly. However, the wave function that we write down may not be unique in some cases; there may be several independent polynomials that have the same pattern of zeros. This corresponds to there being several distinct CFTs whose simple-current algebra possesses the same pattern of zeros. In the following examples, we will make this choice when necessary so that we can explicitly write down a wave function with a certain pattern of zeros.

A. $n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

These states are all Abelian and correspond to the Halperin (m, n, l) states. The unit cell spanned by the $\vec{n}_I$ consists only of the points $(1, 0)$ and $(0, 1)$; By translation invariance $S_{(1,0)} = S_{(0,1)} = 0$. Thus these states are completely characterized by the matrix m and are of the form

$$\Phi = \prod_{i<j}(z_i - z_j)^{m_{11}} \prod_{i<j}(w_i - w_j)^{m_{22}} \prod_{i,j}(z_i - w_j)^{m_{12}}. \tag{77}$$

The m matrix here is exactly the K matrix that describes all Abelian FQH states.[5] We also have $h^{sc}_{\vec{a}} = 0$ for all of these states – the CFT has no simple-current part and corresponds to a $U(1)^2$ Gaussian CFT.

Although the single-layer hierarchy states, such as the $\nu = 2/5$ hierarchy state, do not have ideal single-layer wave functions, there are ideal multilayer states that have the same topological orders as those single-layer hierarchy states. For example, the topological order in the

$\nu = 2/5$ hierarchy state is described by the K-matrix $K = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$ (in the symmetric basis).[4,5] Such a topological order can be represented by the ideal bilayer state with $n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $m = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$. So although the pattern-of-zeros construction does not directly classify those single-layer hierarchy states, their topological orders can still be described by the pattern-of-zeros approach.

B. $n = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$

This choice of n requires that the electron operators in the CFT must take the form

$$V_{e1} = \psi e^{i\sum_J M_{1J}\phi_J},$$
$$V_{e2} = \psi e^{i\sum_J M_{2J}\phi_J}, \tag{78}$$

where $\psi^2 = 1$ and ψ has scaling dimension that is integer or half-integer. This latter fact can be obtained from the condition $\Delta_3((1,0),(1,0),(1,0)) =$ even. One general way of constructing such a simple-current operator is by expressing it as a set of Majorana fermions from several copies of the Ising CFT:

$$\psi = \psi^{(1)} \cdots \psi^{(a)}, \tag{79}$$

where $\psi^{(i)}$ is the Majorana fermion from the ith Ising CFT. Such an operator has scaling dimension $h^{sc}_{\psi} = a/2$ and gives rise to the following FQH wavefunction:

$$\Phi(\{z_i,w_i\}) = \mathrm{Pf}\left(\frac{1}{x_i - x_j}\right)^a \times \Phi_{ga}. \tag{80}$$

x_i represents the coordinates in both layers:

$$x_i \equiv \begin{cases} z_i & 1 \le i \le N_1 \\ w_{i-N_1} & N_1 < i \le N_2 \end{cases} \tag{81}$$

Φ_{ga} is defined in (72). Note that the simple-current algebra in this case implies $\psi^2_{(1,0)} = \psi^2_{(0,1)} = 1$. That is, these states can all also be viewed as satisfying the cluster condition for $n = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, but with a different choice m. For every pattern-of-zeros solution found here, there is an equivalent one for $n = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

Later, we will list the solutions with the $n = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ cluster structure. Some of those solutions actually have $n = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$.

C. $n = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Here, the electron operators must be of the form

$$V_{e1} = \psi e^{i\sum_J M_{1J}\phi_J},$$
$$V_{e2} = e^{i\sum_J M_{2J}\phi_J}. \tag{82}$$

The fact that $\vec{n}_1 = (1,1)$ however also forces $\psi = 1$. Thus in fact the corresponding CFTs do not have a simple-current part; they all correspond to a $U(1)^2$ Guassian CFT. All of these states are therefore Abelian and correspond to the Halperin (m,n,l) states. Thus, all pattern-of-zeros solutions where $n = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ actually also satisfy the cluster condition for $n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Using the cluster condition and the fact that $S_{(1,0)} = S_{(0,1)} = 0$, it is easy to see that the pattern of zeros for these states are completely characterized by the matrix m and therefore that these states are identical to the Halperin states.

D. $n = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

In this case, the electron operators in the CFT must take the form

$$V_{e1} = e^{i\sum_J M_{1J}\phi_J},$$
$$V_{e2} = \psi e^{i\sum_J M_{2J}\phi_J}, \tag{83}$$

where $\psi^2 = 1$. From $\Delta_3((0,1),(0,1),(0,1)) =$ even, it follows that the scaling dimension of ψ is integer or half-integer. This means that we can in general write it as a product of a Majorana fermion operators from a independent copies of the Ising CFT. Thus the pattern-of-zeros solution for this choice of n includes only two classes of states: the Abelian Halperin states and the following version of the Pfaffian states:

$$\Phi(\{z_i,w_i\}) = \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right)^a \times \Phi_{ga} \tag{84}$$

Such a state spontaneously breaks the discrete Z_2 symmetry associated with interchanging the two layers.

E. $n = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$: non-Abelian bilayer states

For this choice of n, the corresponding CFTs have two simple current operators, $\psi_{(0,1)}$ and $\psi_{(1,0)}$, which each square to the identity: $\psi^2_{(1,0)} \sim \psi^2_{(0,1)} \sim 1$. Thus there are a total of three distinct primary currents: $\psi_{(1,0)}$, $\psi_{(0,1)}$, and $\psi_{(1,1)} \equiv \psi_{(1,0)}\psi_{(0,1)}$. The pattern of zeros can be fully specified by specifying the scaling dimensions of

these simple current operators, $h^{sc}_{(1,0)}$, $h^{sc}_{(0,1)}$, and $h^{sc}_{(1,1)}$, and the matrix m, which specifies the $U(1)^2$ part of the electron operators in the CFT.

Applying $\Delta_3(\vec{a},\vec{a},\vec{a}) =$ even for this choice of n shows that the simple-current scaling dimensions are all integer or half integer: $2h^{sc}_{\vec{a}} \in \mathbb{Z}$. One general way of encoding these fusion relations and the associated scaling dimensions is to write the electron operators in the CFT using simple-current operators that consist of multiple copies of the $su(3)_2/u(1)^2$ parafermion CFT. The most general choice for the electron operators can be written in the form

$$\begin{aligned} V_{e;1} &= \psi^{(1)}_{\alpha_1}\cdots\psi^{(a)}_{\alpha_a} e^{i\sum_J M_{1J}\phi_J}, \\ V_{e;2} &= \psi^{(a-c)}_{\beta_1}\cdots\psi^{(a+b-c)}_{\beta_b} e^{i\sum_J M_{2J}\phi_J}, \end{aligned} \tag{85}$$

where $\psi^{(a)}_{\alpha}$ is a simple current from the ath copy of the $su(3)_2/u(1)^2$ parafermion CFT and α_i and β_i refer to either $(1,0)$, $(0,1)$, or $(1,1)$. Some explicit forms for such correlators of simple-current operators in the $su(3)_2/u(1)^2$ parafermion CFT were discussed in Ref. 18. Computing these correlation functions provides one way – not necessarily unique – of constructing a wave function with the desired pattern of zeros.

These pattern-of-zeros solutions can naturally be grouped into two distinct classes. In the first class, $V_{e;1}$ and $V_{e;2}$ contain only one kind of simple-current, either $\psi_{(0,1)}$, $\psi_{(1,0)}$, or $\psi_{(1,1)}$, from each copy of the $su(3)_2/u(1)^2$ CFT. For example, if $\psi^{(i)}_{(1,0)}$ appears in $V_{e;1}$, then $V_{e;2}$ cannot contain $\psi^{(i)}_{(1,1)}$ or $\psi^{(i)}_{(0,1)}$. In such a situation, we can think of $\psi^{(i)}_{\alpha_i}$ as being the Majorana fermion from the Ising CFT. This means that these states can be written in terms of multiple copies of the Ising CFT . This class of multilayer states can be written by choosing

$$\begin{aligned} V_{e;1} &= \psi^{(1)}\cdots\psi^{(a)} e^{i\sum_J M_{1J}\phi_J}, \\ V_{e;2} &= \psi^{(a-c)}\cdots\psi^{(a+b-c)} e^{i\sum_J M_{2J}\phi_J}, \end{aligned} \tag{86}$$

where $\psi^{(a)}$ is the Majorana fermion from the ath copy of the Ising CFT. The wave function for this class of states is therefore:

$$\Phi(\{z_i,w_i\}) = \mathrm{Pf}\left(\frac{1}{z_i-z_j}\right)^{a-c}\mathrm{Pf}\left(\frac{1}{w_i-w_j}\right)^{b-c}\mathrm{Pf}\left(\frac{1}{x_i-x_j}\right)^{c}\prod_{i<j}(z_i-z_j)^{\frac{m_{11}}{2}}\prod_{i<j}(w_i-w_j)^{\frac{m_{22}}{2}}\prod_{i,j}(z_i-w_j)^{\frac{m_{12}}{2}}. \tag{87}$$

x_i represents the coordinates in both layers:

$$x_i \equiv \begin{cases} z_i & 1 \le i \le N_1 \\ w_{i-N_1} & N_1 < i \le N_2 \end{cases} \tag{88}$$

This is the simplest generalization of the single-layer non-Abelian states to a class of non-Abelian bilayer states: the interlayer Pfaffian states. The simplest version of this, with $a = b = c$, is closely related to (and identical to for certain choices of m) the spin-charge separated non-Abelian spin singlet wavefunction proposed in Ref. 24.

The second class of states cannot be written in terms of multiple copies of the Ising CFT – the full $su(3)_2/u(1)^2$ parafermion CFT is necessary. The first class of states, which can be written only using the Ising CFT, have the property that their pattern of zeros satisfies $\Delta_3(\vec{a},\vec{b},\vec{c}) =$ even for all choices of $\vec{a}$, $\vec{b}$, and $\vec{c}$. The second class of states, for which the full $su(3)_2/u(1)^2$ CFT is necessary, satisfies $\Delta_3(\vec{a},\vec{b},\vec{c}) =$ odd for certain choices of $\vec{a}$, $\vec{b}$, and $\vec{c}$.

Let us compare the simple-current algebra of the $su(3)_2/u(1)^2$ parafermion CFT to what one would obtain using two copies of the Ising CFT. If we used two copies of the Ising CFT, we could have $\psi_{(1,0)} = \psi^{(1)}$ with scaling dimension 1/2, $\psi_{(0,1)} = \psi^{(2)}$ with scaling dimension 1/2, and $\psi_{(1,1)} \equiv \psi^{(1)}\psi^{(2)}$ with scaling dimension 1. This satisfies $\Delta_3((1,0),(0,1),(1,1)) =$ even. On the other hand, in the $su(3)_2/u(1)^2$ parafermion CFT, the only difference is that $\psi_{(1,1)}$ also has scaling dimension 1/2. Thus in this latter theory, two fermions combine to give another fermion. This yields $\Delta_3((1,0),(0,1),(1,1)) =$ odd. The fact that there are valid single-valued translationally invariant FQH wavefunctions that arise from unitary CFTs and that have $\Delta_3(\vec{a},\vec{b},\vec{c}) =$ odd for certain choices of $\vec{a}$, $\vec{b}$, and $\vec{c}$ suggests (see Section V E) that we should impose $\Delta_3(\vec{a},\vec{b},\vec{c}) =$ even not in general but only if $\vec{a}$, $\vec{b}$, and $\vec{c}$ are collinear through the origin.

Let us examine the pattern of zeros for a few of the simplest examples of these non-Abelian bilayer states. There is a fermionic $\nu = 2/3$ state with

$$\begin{aligned} m = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \qquad & \{S_{(2,0)} = 0, S_{(1,1)} = 0, S_{(0,2)} = 0\} \\ \nu = 2/3 \qquad & \{h^{sc}_{(1,0)} = \frac{1}{2}, h^{sc}_{(1,1)} = 0, h^{sc}_{(0,1)} = \frac{1}{2}\}. \end{aligned} \tag{89}$$

This is the pattern of zeros for the interlayer Pfaffian

state, which is of the form

$$\Psi(\{z_i, w_i\}) = \mathrm{Pf}\left(\frac{1}{x_i - x_j}\right)\Phi_{(2,2,1)}(\{z_i, w_i\}). \quad (90)$$

We use the notation

$$\Phi_{(\alpha,\beta,\gamma)} = \prod_{i<j}(z_i - z_j)^\alpha \prod_{i<j}(w_i - w_j)^\beta \prod_{i,j}(z_i - w_j)^\gamma. \quad (91)$$

There are also fermionic states at $\nu = 4/5$ and $\nu = 4/7$. These have the following pattern of zeros:

$$m = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \qquad \{S_{(2,0)} = 0, S_{(1,1)} = 0, S_{(0,2)} = 0\}$$
$$\nu = 4/5 \qquad \{h^{sc}_{(1,0)} = \frac{1}{2}, h^{sc}_{(1,1)} = \frac{1}{2}, h^{sc}_{(0,1)} = \frac{1}{2}\}. \quad (92)$$

$$m = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} \qquad \{S_{(2,0)} = 0, S_{(1,1)} = 1, S_{(0,2)} = 0\}$$
$$\nu = 4/7 \qquad \{h^{sc}_{(1,0)} = \frac{1}{2}, h^{sc}_{(1,1)} = \frac{1}{2}, h^{sc}_{(0,1)} = \frac{1}{2}\}. \quad (93)$$

The state at $\nu = 4/7$ is the non-Abelian spin singlet state that was proposed in Ref. 17.

Note once again that the pattern of zeros m and $\{S_{\vec{a}}\}$ refer to the pattern of zeros of the symmetric polynomial, $\Phi = \frac{\Psi}{\prod_{I;i<j}(z_i^I - z_j^I)}$.

X. DISCUSSION OF RESULTS AND RELATION TO EXPERIMENT

In single-layer quantum Hall samples, a quantum Hall plateau is seen at $\nu = 5/2$, but not at $\nu = 1/2$. The reason is that even though in all of these cases there is a single half-filled Landau level, the existence of the two filled extra Landau levels modifies the effective interactions between the electrons in the unfilled level. In the $\nu = 5/2$ case, numerical calculations suggest that the these effective interactions are modified in such a way that a non-Abelian quantum Hall state may be realized.

Experiments on multicomponent quantum Hall systems should be able to probe an even wider variety of regimes with distinct effective interactions. For example, for a two-component FQH system, we can study systems in which the spin degree of freedom is present, two-dimensional electron systems with two quantum wells, wide single-layer systems in which the electrons spontaneously form a double-layer system due to Coulomb repulsion, or systems in which there may be two valleys for the free quasiparticle spectrum (such as in graphene or SiGe heterostructures), etc. In many of these cases, experimentalists can also tune to some extent the degree of correlation between the two components. For example, in double layer systems, application of a parallel magnetic field can tune the tunneling and correlation between the layers. There may also be some degree of tunability in the relative densities between the two components in addition to being able to probe FQH states with different numbers of filled Landau levels. With this greatly increased amount of variability and tunability in the effective interactions between electrons in the unfilled Landau levels, it is possible that a non-Abelian state can be realized in a two-component quantum Hall system.

Since the pattern of zeros provides a systematic classification and characterization of a wide variety of quantum Hall states, it provides us with a general sense of how all of the non-Abelian bilayer states are related and which ones are simpler than other ones. Just as we know that the single-layer Pfaffian quantum Hall state is the simplest non-Abelian generalization of the Laughlin states, we can determine the simplest non-Abelian generalization of the Halperin bilayer states and therefore single out some of the possibilities that may be experimentally viable.

In Ref. 25, we have given an overview of some of the simplest non-Abelian bilayer states that we find and that occur at filling fractions at which experiments on two-component FQH systems have already observed incompressible states. Here we briefly summarize that discussion and supplement details of the calculations of various topological properties of the candidate states.

Experiments have so far observed FQH plateaus in two-component systems at $\nu = 2/3$, $4/5$, $4/7$, $4/9$, $6/5$, $6/7$, $1/4$, etc.[26–31] In some cases, these plateaus have been observed in both bilayer and spin-unpolarized single-layer systems, while in others, the plateau has only been observed in one of them. At all of these filling fractions, there exists also one (or several) candidate Abelian phase(s); in most cases, it is assumed that these plateaus are described by one of the Abelian phases. However, the pattern-of-zeros construction also yields many simple non-Abelian states at these filling fractions. In some situations, we expect the non-Abelian states to be good candidate states.

There are at least two dimensionless quantities that are important determining factors for which FQH state is realized. The first parameter is $\alpha \equiv V_{\mathrm{inter}}/V_{\mathrm{intra}}$, where V_{inter} is the potential for interlayer repulsion and V_{intra} is the potential for intralayer repulsion. The second parameter is $\gamma \equiv t/V_{\mathrm{intra}}$, where t is the interlayer hopping amplitude. In the limit $\alpha \sim 0$ and $\gamma \sim 0$, the system will be a FQH state that consists of two independent single-layer FQH states in each layer. In the limit $\gamma \gg 1$ and $\alpha \sim 0$, a single-layer FQH state may be observed. But if we keep $\gamma \sim 0$ and increase α from $\alpha \sim 0$, then the FQH state formed by two independent single-layer FQH states in each layer must undergo a phase transition into either a compressible phase or a new incompressible state. In the latter case, an Abelian hierarchy state (such as a bilayer composite fermion state) may form, which would in most cases be a state described by a 4×4 or more complicated

K-matrix and would have four or more edge modes. The other possibility is that a non-Abelian two-component state may form. The pattern-of-zeros construction yields non-Abelian two-component wave functions that have zeros when particles from different layers approach each other, indicating that they can accomodate situations in which $\alpha \sim 1$. Additionally, these states generally have less than 4 edge modes; if we use the number of edge modes as a measure of the complexity of the state, then the non-Abelian states are simpler and may therefore be realized experimentally.

At $\nu = 2/3$, experiments on wide single quantum wells have observed a phase transition from a bilayer to single-layer state while experiments on single-layer systems have seen a phase transition from a spin-polarized to a spin-unpolarized state. In the limit $\alpha \sim 0$ and $\gamma \sim 0$, the system should be in the $(3,3,0)$ state. As α is increased while $\gamma \sim 0$, one possibility is the $(1,1,2)$ state. This wave function appears unphysical, because it has higher order zeros as particles from different layers approach each other than particles from within the same layer. Another wave function, which has the same topological order as $(1,1,2)$, is a spin-singlet composite fermion state.[32] There are two other plausible non-Abelian states in this situation. One is the following interlayer Pfaffian state (see (89) and (90)):

$$\Psi_{2/3|\text{inter}} = \text{Pf}\left(\frac{1}{x_i - x_j}\right)\Phi_{(2,2,1)}(\{z_i, w_i\}). \qquad (94)$$

The other is the following intralayer Pfaffian state

$$\Psi_{2/3|\text{intra}} = \text{Pf}\left(\frac{1}{z_i - z_j}\right)\ \text{Pf}\left(\frac{1}{w_i - w_j}\right)\Phi_{(2,2,1)}(\{z_i, w_i\}), \qquad (95)$$

which has even higher order zeros as particles from different layers approach each other. $\Psi_{2/3|\text{intra}}$ has a cluster structure $n = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ while $\Psi_{2/3|\text{inter}}$ has a cluster structure $n = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. At $\nu = 2/3$ there are also two single-layer possibilities that may be realized as γ is increased. These are the particle-hole conjugate of the $\nu = 1/3$ Laughlin state and the Z_4 parafermion Read-Rezayi state.

At $\nu = 4/5$, $4/7$, and $4/9$, we have the following non-Abelian states (see (92) and (93)):

$$\begin{aligned} \Psi_{4/5} &= \Phi_{sc}(\{z_i, w_i\})\Phi_{(2,2,\frac{1}{2})}(\{z_i, w_i\}), \\ \Psi_{4/7} &= \Phi_{sc}(\{z_i, w_i\})\Phi_{(2,2,\frac{3}{2})}(\{z_i, w_i\}), \\ \Psi_{4/9} &= \Phi_{sc}(\{z_i, w_i\})\Phi_{(4,4,\frac{1}{2})}(\{z_i, w_i\}), \end{aligned} \qquad (96)$$

where $\Phi_{sc} = \langle \prod_i \psi_1(z_i)\psi_2(w_i)\rangle$ is a correlation function in the $su(3)_2/u(1)^2$ parafermion CFT. These states all have $2\frac{6}{5}$ edge modes.

ν	Proposed States	Edge Modes	Shift $\mathcal{S}$
2/3	$(3,3,0)$	2	3
	$(1,1,2)$	2	1
	$2/3\|_{\text{inter}}$ (see eqn. (94))	$2\frac{1}{2}$	3
	$2/3\|_{\text{intra}}$ (see eqn. (95))	3	3
	Z_4 parafermion	3	3
	P-H conjugate of $\nu = 1/3$	$1_R + 1_L$	0
4/5	$(2/5, 2/5\|0)$	4	4
	$su(3)_2/u(1)^2$ (see eqn. (96))	$2\frac{6}{5}$	3
	$(2/3, 2/3\|1)$	$2_R + 2_L$	0
4/7	$(2/7, 2/7\|0)$	4	2
	$su(3)_2/u(1)^2$ (see eqn. (96))	$2\frac{6}{5}$	3
	$(2/5, 2/5\|1)$	4	4
	$(2/3, 2/3\|2)$	$1_R + 3_L$	0
1/4	$(5,5,3)$	2	5
	$(7,7,1)$	2	7
	Inter-layer Pfaffian (see eqn. (98))	$2\frac{1}{2}$	7
	Single-layer Pfaffian	$1\frac{1}{2}$	5

TABLE I: Proposed explanations for incompressible states at experimentally relevant filling fractions, $\nu = 2/3$, $4/5$, $4/7$, and $1/4$, in two-component FQH systems. The bilayer composite fermion state $(\nu_1, \nu_2|m)$[33] refers to the state $\prod_{i,j}(z_i - w_j)^m \Phi_{\nu_1}(\{z_i\})\Phi_{\nu_2}(\{w_i\})$, where Φ_ν is a single layer composite fermion state at filling fraction ν. For $(2/3, 2/3|m)$, we have taken the single layer $2/3$ state to be the particle-hole conjugate of the Laughlin state. $n_R + n_L$ indicates that there are n_R right-moving edge modes and n_L left-moving edge modes. See Appendix C for details of how to calculate the number of edge modes and the shift $\mathcal{S}$.

The other set of proposed Abelian states are the bilayer composite fermion states[33] $(\nu_0, \nu_0|m)$, which refer to the wave function

$$\Phi_{(\nu_0,\nu_0|m)} = \prod_{i,j}(z_i - w_j)^m \Phi_{\nu_0}(\{z_i\})\Phi_{\nu_0}(\{w_i\}). \qquad (97)$$

Here $\Phi_{\nu_0}(\{z_i\})$ is a single-layer FQH state at filling fraction ν_0. These states have 4 edge modes, indicating that they may be less stable than the alternative non-Abelian possibilities.

Recently, an incompressible state was found at $\nu = 1/4$ and it is unclear what phase this corresponds to and even whether it is a single-layer or double-layer phase.[34] Some possibilites that have recently been considered[35] are the $(5,5,3)$ and $(7,7,1)$ Halperin states and the $\nu = 1/4$ single-layer Pfaffian. The pattern-of-zeros construction yields many other alternative possibilities, perhaps the most physical (and simplest) of which is the following interlayer Pfaffian:

$$\Psi(\{z_i, w_i\}) = \text{Pf}\left(\frac{1}{x_i - x_j}\right)\Phi_{(6,6,2)}(\{z_i, w_i\}). \qquad (98)$$

In Table I we summarize some of the filling fractions at which incompressible states have been experimentally observed in two-component FQH systems. For each filling

Quasiparticles for Interlayer Pfaffian at $\nu = 2/3$		
CFT Operator	Total Charge	Scaling Dimension
$V_{e1} = \psi e^{i\sqrt{3/2}\phi_+ + i\sqrt{\frac{1}{2}}\phi_-}$	1	3/2
$V_{e2} = \psi e^{i\sqrt{3/2}\phi_+ - i\sqrt{\frac{1}{2}}\phi_-}$	1	3/2
$e^{i\frac{2}{3}\sqrt{\frac{3}{2}}\phi_-}$	2/3	1/3
$e^{i\frac{1}{3}\sqrt{\frac{3}{2}}\phi_+ + i\frac{1}{\sqrt{2}}\phi_-}$	1/3	1/3
$\sigma e^{i\frac{1}{3}\sqrt{\frac{3}{2}}\phi_-}$	1/3	7/48
$\sigma e^{i\frac{2}{3}\sqrt{\frac{3}{2}}\phi_+ + i\sqrt{\frac{1}{2}}\varphi_-}$	2/3	31/48
$\sigma e^{i\frac{1}{\sqrt{2}}\phi_-}$	0	5/16
ψ	0	1/2
$\psi e^{i\frac{2}{3}\sqrt{\frac{3}{2}}\phi_+}$	2/3	5/6
$\psi e^{i\frac{1}{3}\sqrt{\frac{3}{2}}\phi_+ + i\frac{1}{\sqrt{2}}\phi_-}$	1/3	5/6

TABLE II: Quasiparticle operators from the CFT for $\nu = 2/3$ Interlayer Pfaffian states. The scalar boson fields ϕ_+ and ϕ_- are related to the total and relative density fluctuations of the two layers, respectively. σ is the spin field in the Ising CFT, which has scaling dimension 1/16.

ν	Charge q_{min}	Scaling Dimension h
$2/3\|_{\text{inter}}$	1/3	$\frac{1}{16} + \frac{1}{12} + 0$
$2/3\|_{\text{intra}}$	1/6	$\frac{1}{16} + \frac{1}{48} + \frac{1}{16}$
4/5	1/5	$\frac{1}{10} + \frac{1}{40} + \frac{1}{24}$
4/7	1/7	$\frac{1}{10} + \frac{1}{56} + \frac{1}{8}$
4/9	1/9	$\frac{1}{10} + \frac{1}{72} + \frac{1}{56}$
1/4	1/8	$\frac{1}{16} + \frac{1}{32} + 0$

TABLE III: Charge and scaling dimensions of the quasiparticle operators with minimal nonzero total charge in the non-Abelian bilayer states discussed here. In the scaling dimension, the first term comes from the non-Abelian part, the second term comes from the total density fluctuations (the $U(1)$ part), the third term comes from the relative density fluctuations of the two layers (also the $U(1)$ part).

fraction we list some of the proposed wave functions that may characterize the topological order of those phases, the number of edge modes, and their respective shifts on the sphere. We list the qasiparticles, their electric charges, and their scaling dimensions for the interlayer Pfaffian state at $\nu = 2/3$ in Table II. In Table III, we list the quasiparticles with the minimal electric charge and their scaling dimensions h for the non-Abelian FQH states discussed in this paper [see eqns. (90), (92), (93), (96), (98)]. Those minimally charged quasiparticles may dominate interedge tunneling and give rise to the following I-V curve: $I \propto V^{4h-1}$ in the $T = 0$ limit.

In summary, we find many simple non-Abelian bilayer states that occur at experimentally observed filling fractions. For certain effective interactions among the electrons in the unfilled Landau levels, these states may be more favorable than their Abelian counterparts. In these cases, the non-Abelian states have larger interlayer correlations and therefore may be energetically more favorable

$\nu = 2/3$ Interlayer Pfaffian						
		N_1				
		2	4	6	8	10
N_2	2	(3,3)	(7,5)	(11, 7)	(15,9)	(19,11)
	4	(5,7)	(9,9)	(13,11)	(17,13)	(21,15)
	6	(7,11)	(11,13)	(15,15)	(19,17)	(23, 19)
	8	(9,15)	(13,17)	(17,19)	(21,21)	(25,23)
	10	(11,19)	(15,21)	(19, 23)	(23,25)	(27,27)

TABLE IV: Values of (N_Φ^1, N_Φ^2) that yield rotationally invariant states on the sphere for various choices of (N_1, N_2) for the $\nu = 2/3$ interlayer Pfaffian.

$\nu = 4/5$ $su(3)_2/u(1)^2$ parafermion state						
		N_1				
		2	4	6	8	10
N_2	2	(2,2)	(6,3)	(10, 4)	(14,5)	(18,6)
	4	(3,6)	(7,7)	(11,8)	(15,9)	(19,10)
	6	(4,10)	(8,11)	(12,12)	(6, 13)	(20, 14)
	8	(5,14)	(9,15)	(13,6)	(17,17)	(21,18)
	10	(6,18)	(10,19)	(14, 20)	(18,21)	(22,22)

TABLE V: Values of (N_Φ^1, N_Φ^2) that yield rotationally invariant states on the sphere for various choices of (N_1, N_2) for the $\nu = 4/5$ $su(3)_2/u(1)^2$ parafermion state.

in situations in which the interlayer repulsion is comparable to the intralayer repulsion.

A. Conditions on Filling the Sphere

A useful tool for identifying FQH states in numerical studies of exact diagonalization on finite systems on a sphere is to look at what values of the shift, $\mathcal{S} = \nu^{-1} N_e - N_\Phi$, a ground state with zero total angular momentum is found. This then limits the possibilities of which topological phase is realized in the system to those that have that particular value of the shift. Similarly, in such numerical studies of multilayer systems, one can look for the different sets $(N_1, \cdots, N_f; N_\Phi^1, \cdots, N_\Phi^{N_f})$ that yield a ground state with zero total angular momentum. Each topological phase will have its own list of $(N_1, \cdots, N_f; N_\Phi^1, \cdots, N_\Phi^{N_f})$ that let it fill the sphere; analyzing this can be a useful way of determining which topological phase is obtained numerically. In Section V D, we found conditions that $\vec{N}$ and $\vec{N}_\Phi$ should satisfy for the FQH state to fill the sphere.

For states that have a cluster structure $n = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, we find that the condition (46) becomes trivial as long as $\vec{N} = \sum_I k_I \vec{n}_I$, where k_I is an integer. This means that as long as N_1 and N_2 are even and N_Φ^1, N_Φ^2 satisfy (44), then these states can fill the sphere. In this case, we find

that (44) reduces to the form

$$\begin{pmatrix} N_\Phi^1 + \mathcal{S} \\ N_\Phi^2 + \mathcal{S} \end{pmatrix} = M \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}, \tag{99}$$

where M is a 2×2 matrix and $\mathcal{S}$ is the shift, which can be calculated using eqn. (C11). The states that we have been considering are of the form $\Phi = \Phi_{sc} \times \Phi_{(\alpha,\beta,\gamma)}$, for which $M = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$. Tables IV and V lists some examples.

XI. SUMMARY

In this paper, we generalized the pattern-of-zeros characterization and classification of FQH states to multicomponent cases. We found that the topological orders in a multicomponent FQH state can be characterized by the following data: a matrix n that describes the cluster structure, a matrix m and a sequence $\{S_{\vec{a}}\}$ that describes the pattern of zeros.

Our pattern-of-zeros characterization gives us a general quantitative view on a large class of Abelian and non-Abelian bilayer FQH states, which allow us to determine which states are simpler than other states. We find some simplest non-Abelian generalizations of the Laughlin-Halperin Abelian bilayer states. Those simple non-Abelian states may describe some of the bilayer/spin-unpolarized FQH states observed in experiments and numerical calculations.

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APPENDIX A: OCCUPATION NUMBER CHARACTERIZATION

In the single-component pattern-of-zeros description, there is an occupation number characterization that is a useful way to understand both the ground states and the quasiparticles in FQH states. The generalization to multilayer states does not appear to be quite as simple or useful, but for the sake of completeness we will analyze it below.

A convenient set of single-particle basis states for particles in the lowest Landau level are monomials of the form z^m, for integer m. Thus, a basis for symmetrized wavefunctions of N_I particles of type I is given by:

$$\Phi_{\{\vec{n_l}\}} = \sum_{\{P_I\}} \prod_I \prod_{i=1}^{N_I} (z^I_{P_I(i)})^{l_i^I}, \tag{A1}$$

where P_I is a permutation of the particles of type I, l_i^I is an integer, and $\vec{n}_l$ is a vector whose component n_l^I is the number of particles of type I that occupy the l^{th} orbit. The polynomial Φ that we are interested in can be expanded in terms of these basis states as:

$$\Phi = \sum_{\{\vec{n}_l\}} C_{\{\vec{n}_l\}} \Phi_{\{\vec{n}_l\}}. \tag{A2}$$

Now we may ask what kind of boson occupations $\{\vec{n}_l\}$ will be present in the sum (A2) for a polynomial Φ with a given pattern-of-zeros $\{S_{\vec{a}}\}$. To answer this question, let us set $z_1^1 = 0$ in $\Phi(\{z_i^I\})$. Since Φ is nonzero when $z_1^1 = 0$ due to translation invariance, there must be a boson occupation $\{\vec{n}_l\}$ in the above sum that contains at least one boson occupying the $(z^1)^{l=0}$ orbital. That is, there is a term in the above sum with $n_0^1 > 0$. Now, suppose that we bring a second particle of the same type, z_2^1 to 0. The minimal power of z_2^1 in $\Phi(0, z_2^1, \cdots)$ is $D_{\vec{e}_1,\vec{e}_1}$:

$$\begin{aligned}\Phi(0, z_2^1, z_3^1, ..., z_{N_M}^M) \sim & (z_2^1)^{D_{\vec{e}_1,\vec{e}_1}} P_2(z_3^1, z_4^1, ...) \\ & + O((z_2^1)^{D_{\vec{e}_1,\vec{e}_1}+1}).\end{aligned} \tag{A3}$$

Thus, among those $\{\vec{n}_l\}$ that have at least one boson of type 1 occupying the $(z^1)^{l=0}$ orbital, there must also be an $\{\vec{n}_l\}$ that contains a second boson of type 1 occupying the $(z^1)^{l_{2\vec{e}_1}}$ orbital where $l_{2\vec{e}_1} = D_{\vec{e}_1,\vec{e}_1} = S_{2\vec{e}_1} - S_{\vec{e}_1}$. Next, assume that two bosons occupy the $(z^1)^0$ and $(z^1)^{l_{2\vec{e}_1}}$ orbital, and bring a third particle of type 1 to 0; the minimal power of z_3^1 is $D_{2\vec{e}_1,\vec{e}_1}$:

$$\begin{aligned}P_2(z_3^1, z_4^1, ...) \sim & (z_3^1)^{D_{2\vec{e}_1,\vec{e}_1}} P_3(z_4^1, z_5^1, ...) \\ & + O((z_3^1)^{D_{2\vec{e}_1,\vec{e}_1}+1}).\end{aligned} \tag{A4}$$

Thus, among those $\{\vec{n}_l\}$ that have two type 1 bosons occupying the $l = 0$ and $l_{2\vec{e}_1}$ orbitals, there is a third boson of type 1 occupying the $l_{3\vec{e}_1} = D_{2\vec{e}_1,\vec{e}_1} = S_{3\vec{e}_1} - S_{2\vec{e}_1}$ orbital. Continuing in this way, we see that there must be a type 1 boson occupying the orbitals $l^1_{a\vec{e}_1} = S_{a\vec{e}_1} - S_{(a-1)\vec{e}_1}$ for $a = 1, \cdots, N_1$. After taking all the type 1 particles to 0, we may begin to take the type 2 particles to zero, one by one, thus obtaining that there must be a type 2 boson occupying the orbitals $l^2_{a\vec{e}_2+N_1\vec{e}_1} = S_{a\vec{e}_2+N_1\vec{e}_1} - S_{(a-1)\vec{e}_2+N_1\vec{e}_1}$. Continuing this argument for bosons of every type, we find that there must be a term in the sum (A2) with occupation number described by the above sequence of $l^I_{\vec{a}}$'s.

However, in the above argument, we chose a particular sequence in which to take various particles to zero. We first took all of the type 1 particles to zero one-by-one, and then all of the type 2 ones, and so on. But we could just as well have made the argument with any sequence. Suppose that after taking i particles to the origin, there is a $\vec{a}_i$-cluster at the origin. Thus $\{\vec{a}_i\}$ is a sequence that describes the order in which we take particles to the origin until all particles are at the origin. For every such sequence, we may make the above argument and argue that if $\vec{a}_{i+1} = \vec{a}_i + \vec{e}_I$, then there must be a type I boson occupying the orbital $l^I_{\vec{a}_{i+1}} = S_{\vec{a}_i+\vec{e}_I} - S_{\vec{a}_i}$. If we enumerate all the different sequences $\{\vec{a}_i\}$ by an integer α, then by considering each α, we see that there must

be a term in the sum (A2) with occupation number $\vec{n}_l^\alpha$. $n_l^{I;\alpha}$ would be the number of i, along the sequence $\{\vec{a}_i\}$, for which $l_i^I = l$. Notice that $l_{\vec{a}}^I$ must be non-zero. Thus we have the following important condition on $S_{\vec{a}}$:

$$l_{\vec{a}}^I = S_{\vec{a}} - S_{\vec{a}-\vec{e}_I} \geq 0. \tag{A5}$$

The analysis above can be thought of in the following way. Consider an N_f-dimensional lattice $\mathbb{Z}^{N_f}$, where N_f is the number of layers. At every site $\vec{a}$ of this lattice ($a_I \geq 0$) we can associate the nonnegative integer $S_{\vec{a}}$. On each link $(\vec{a}, \vec{a} - \vec{e}_I)$, of the lattice we may also associate an integer $l_{\vec{a}}^I = S_{\vec{a}} - S_{\vec{a}-\vec{e}_I}$. Now consider any directed path from the origin to $\vec{N}$ (N_I is the number of particles of type I), in which the sum of the coordinates of every point on the path is one larger than the sum of the coordinates of the point preceding it, and enumerate the set of these paths by α. To each such path we associate an occupation number sequence $\vec{n}_l^\alpha$, where $n_l^{I;\alpha}$ is the number of links along the path α whose $l_{\vec{a}}^I = l$. If Φ has a pattern-of-zeros $\{S_{\vec{a}}\}$, then its basis expansion (A2) must contain a term with occupation number $\vec{n}_l^\alpha$. Thus we may rewrite (A2) as

$$\Phi = \sum_\alpha C_\alpha \Phi_{\vec{n}_l^\alpha} + \sum_{\{\vec{\tilde{n}}_l\}} D_{\{\vec{\tilde{n}}_l\}} \Phi_{(\{\vec{\tilde{n}}_l\})}(\{z_i^I\}). \tag{A6}$$

The two sequences $\{S_{\vec{a}}\}$ and $\{\vec{n}_l^\alpha\}$ contain the same information and are one-to-one labellings of each other. However, $\{\vec{n}_l^\alpha\}$ is redundant in the sense that it does not need to be specified for every α in order to reconstruct $\vec{l}_{\vec{a}}$. The $\vec{\tilde{n}}_l$ that appear in the second sum characterize the subleading terms that appear when coordinates are brought together; thus those $\{\vec{\tilde{n}}_l\}$ correspond to sequences $\{\tilde{S}_{\vec{a}}\}$ where $\tilde{S}_{\vec{a}} \geq S_{\vec{a}}$.

In the single-layer case, $\{l_a\}$ naturally defined an occupation number sequence $\{n_l\}$, which also described the FQH state in the thin-cylinder limit. In the multi-component generalization, we have $\{\vec{l}_{\vec{a}}\}$, which seems to admit no simple generalization of the above occupation number sequence. Instead, one has such occupation number distributions for a large number of sequences which we enumerated above by α. We have not analyzed on general grounds which particular sequences contribute the most weight to the wavefunction in the thin cylinder limit.

APPENDIX B: $su(3)_2/u(1)^2$ PARAFERMION CONFORMAL FIELD THEORY

Some of the simplest non-Abelian bilayer states are closely related to $su(3)_2/u(1)^2$ parafermion CFT. This CFT has central charge $c = 6/5$ and has three simple currents, ψ_α, ψ_β, and $\psi_{\alpha+\beta} = \psi_\alpha\psi_\beta$, all of which square to the identity and have scaling dimension 1/2.

There are four other primary fields, which are associated with the fundamental representation of $su(3)$. Their scaling dimensions are listed in Table VI. The fusion rules for these fields all follow from the following fusion rule:

Primary fields in $su(3)_2/u(1)^2$	
CFT Operator	Scaling Dimension
σ	1/10
$\sigma_\alpha = \psi_\alpha\sigma$	1/10
$\sigma_\beta = \psi_\beta\sigma$	6/10
$\sigma_{\alpha+\beta} = \psi_{\alpha+\beta}\sigma$	1/10

TABLE VI: Primary fields and their scaling dimensions in the $su(3)_2/u(1)^2$ parafermion CFT.

$$\sigma \times \sigma = 1 + \sigma_\beta. \tag{B1}$$

APPENDIX C: CALCULATIONS FOR CANDIDATE STATES

1. Number of Edge Modes

The total number of edge modes is equal to the central charge of the corresponding CFT for the states that are described by the pattern of zeros. For the hierarchy states, the number of edge modes is given by the rank of the K-matrix. Furthermore, in the latter case, the number of right (left) -moving edge modes is given by the number positive (negative) eigenvalues of the K-matrix.

The interlayer Pfaffian states are described by a CFT that consists of the Ising CFT, with $c = 1/2$, and two scalar boson CFTs, each with $c = 1$. Thus the number of edge modes for the interlayer Pfaffian is $2\frac{1}{2}$.

The intralayer Pfaffian states have two Ising CFTs in addition to the two scalar boson CFTs, so the total number of edge modes is 3.

The central charge of the $su(3)_2/u(1)^2$ parafermion CFT is $c = 6/5$; the two-component FQH states based on this are described by the $su(3)_2/u(1)^2$ theory and two scalar bosons, for a total of $2\frac{6}{5}$ edge modes.

For the (m, m, l) states, the K-matrix is $K = \begin{pmatrix} m & l \\ l & m \end{pmatrix}$. These states have 2 edge modes; if $m > l$, all edge modes move in the same direction; if $m < l$, then there is one right-moving and one left-moving edge mode.

For the states $(\nu_0, \nu_0|m)$, the K matrix is

$$K = \begin{pmatrix} K_{11}^0 & K_{12}^0 & m & 0 \\ K_{21}^0 & K_{22}^0 & 0 & 0 \\ m & 0 & K_{11}^0 & K_{12}^0 \\ 0 & 0 & K_{21}^0 & K_{22}^0 \end{pmatrix}, \tag{C1}$$

where K^0 is the K-matrix in the hierarchical basis of the state Φ_{ν_0}. For the $\nu = 2/5$ state, K^0 in the hierarchical

basis is

$$K^0 = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}. \quad \text{(C2)}$$

For the $\nu = 2/7$ state, K^0 in the hierarchical basis is

$$K^0 = \begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix}. \quad \text{(C3)}$$

For the $\nu = 2/3$ P-H conjugate of the 1/3 Laughlin state, K^0 in the hierarchical basis is

$$K^0 = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}. \quad \text{(C4)}$$

Using this, we find that the P-H conjugate of the 1/3 Laughlin state has edge modes $1_R + 1_L$, $(2/3, 2/3|1)$ has edge modes $2_R + 2_L$, and $(2/3, 2/3|2)$ has $1_R + 3_L$.

2. Shifts on Sphere

In the hierarchy basis, the formula for the shift is given by[36]

$$\mathcal{S} = \frac{1}{\nu} \sum_I (K^{-1})_{1I} K_{II}. \quad \text{(C5)}$$

Using this formula, we find $\mathcal{S} = 0$ for the particle-hole conjugate of the $\nu = 1/3$ Laughlin state.

Now consider the bilayer composite fermion state $(\nu_0, \nu_0|m)$:

$$\Phi_{(\nu_0,\nu_0|m)} = \prod_{i,j} (z_i - w_j)^m \Phi_{\nu_0}(\{z_i\}) \Phi_{\nu_0}(\{w_i\}). \quad \text{(C6)}$$

Let N_Φ^0 be the maximum power of z_1 in $\Phi_{\nu_0}(\{z_i\})$. It satisfies:

$$N_\Phi^0 = \nu_0^{-1} N_1 - \mathcal{S}_0, \quad \text{(C7)}$$

where $\mathcal{S}_0$ is the shift of the state Φ_{ν_0}. The factor $\prod_{i,j}(z_i - w_j)^m$ increases the power of z_1 by mN_2. Thus the maximum power of z_1 in $\Phi_{(\nu_0,\nu_0|m)}$ is N_Φ^1:

$$N_\Phi^1 = \nu_0^{-1} N_1 + mN_2 - \mathcal{S}_0. \quad \text{(C8)}$$

In our cases, $N_1 = N_2$, and the number of flux quanta is the same in each layer, so

$$N_\Phi^1 = (\nu_0^{-1} + m) N_1 - \mathcal{S}_0, \quad \text{(C9)}$$

$\nu_1^{-1} = \nu_0^{-1} + m$ is the filling fraction in one layer. Thus we see that the shift of $\Phi_{(\nu_0,\nu_0|m)}$ is also $\mathcal{S}_0$.

For $(2/3, 2/3|m)$, we take $\Phi_{2/3}$ to be the particle-hole conjugate of the 1/3 Laughlin state. $\Phi_{2/3}$ has shift $\mathcal{S}_0 = 0$. Thus $(2/3, 2/3|m)$ also has shift $\mathcal{S} = 0$. $(2/5, 2/5|m)$ has shift 4, because $\Phi_{2/5}$ has shift 4.

For the Halperin (m, n, l) states, the K-matrix can be written as $K = \begin{pmatrix} m & l \\ l & n \end{pmatrix}$. In this basis, the shift is given by

$$\mathcal{S} = \nu^{-1} \sum_{IJ} K_{IJ}^{-1} K_{JJ}. \quad \text{(C10)}$$

For the states described by the pattern of zeros, we can use the following formula

$$\mathcal{S} = \begin{cases} \nu^{-1} \sum_I \nu_I (m_{II} - S_{\vec{n}_I} + S_{\vec{n}_I - \vec{e}_I}) & \text{for bosons} \\ \nu^{-1} \sum_I \nu_I (m_{II} + 1 - S_{\vec{n}_I} + S_{\vec{n}_I - \vec{e}_I}) & \text{for fermions} \end{cases} \quad \text{(C11)}$$

3. Electron and Quasiparticle Operators for $su(3)_2/u(1)^2$ States

The electron operators for the $su(3)_2/u(1)^2$ FQH states that we discuss are of the form:

$$V_{e1} = \psi_\alpha e^{i\sqrt{\frac{1}{\nu}}\phi_+ + is\phi_-},$$
$$V_{e2} = \psi_\beta e^{i\sqrt{\frac{1}{\nu}}\phi_+ - is\phi_-}, \quad \text{(C12)}$$

where $s = \sqrt{3}/2$, $1/2$, and $\sqrt{7}/2$ for the $\nu = 4/5$, $4/7$ and $4/9$ states, respectively. The quasiparticle operators with minimal total charge are of the form:

$$V_{qp} = \sigma e^{iQ\sqrt{\frac{1}{\nu}}\phi_+ + is_{qp}s\phi_-}, \quad \text{(C13)}$$

and have scaling dimension $h_{qp} = \frac{1}{10} + \frac{Q^2}{2\nu} + \frac{(s_{qp}s)^2}{2}$. The total charge of the quasiparticle is $Q = 1/5$, $1/7$, and $1/9$ for the $\nu = 4/5$, $4/7$ and $4/9$ states, respectively. $s_{qp} = 1/3$, 1, and $1/7$, respectively, for these states.

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